THE COHOMOLOGY OF LIE RINGS

BY

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In this paper we will give an exposition of a cohomology theory for Lie rings (more exactly Lie d-rings, see below for definition) with coefficients in a module. While this concept arose naturally from investigating the problem of "lifting" a transformation group from the base space of a fiber bundle to the total space along lines suggested by recent work of Dr. T. E. Stewart, we shall take up these applications elsewhere and be concerned here with only the formal parts of the theory and an elementary but perhaps a little surprising application to differential geometry which we now explain. Since a differentiable manifold can be canonically reconstructed from its set of differentiable real valued functions, considered as an abstract commutative ring, it follows that any canonical construction C which assigns to each differentiable manifold M a structure $C(M)$ of some sort, gives rise to a construction C' which assigns to each commutative ring R which is isomorphic to the ring of differentiable functions on some manifold (say M) a structure $C'(R) = C(M)$. We may then ask when C' is ring theoretic, i.e. when there is a canonical construction defined for all commutative rings which restricts to C'. The application to differential geometry mentioned above is the statement that if C assigns to M its real cohomology (considered as a graded group) then C' is ring theoretic in the above sense. In the more precise language of categories we will define a functor from the category of commutative rings (with isomorphisms as maps) to the category of graded abelian groups (with isomorphisms as maps) which assigns to the differentiable functions on M the real cohomology of M.[2]

Another perhaps worthwhile facet of the theory is that the cohomology of Lie algebras [**1**] and the de Rham cohomology of a differentiable manifold turn out to be special cases of the same thing. In view of the obvious formal similarities of the two theories it is no surprise that this should be so, but as far as the author knows there has been until now no unified treatment of the two.

Whatever originality can be claimed for this paper does not go beyond the definition of a Lie d-ring and a module over such. Once these definitions are made the further development is completely parallel to that found in [**1**] and only an occasional proof must be changed.

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[2] *Added in proof.* In Abstract 575, Bull. Amer. Math. Soc. July, 1957, A. Nijenhuis defined a functor from the category of commutative real algebras to the category of anti-commutative, graded real algebras which assigns to the algebra of differentiable real valued functors on a differentiable manifold M the real cohomology algebra of M.

1. Basic definitions. In all that follows Γ will be a fixed commutative ring. However, *in starred sections we shall assume that* Γ *includes as a subring the field of rational numbers.* By a Γ-ring we shall mean a commutative ring R whose additive group is a Γ-module in such a way that $\gamma(rs) = r(\gamma s)$ for $\gamma \in \Gamma$ and $r, s \in R$. If Γ is the ring of integers then this is just the usual notion of a commutative ring and if Γ is a field then R is just a commutative algebra over Γ. We define a Γ-subring of R in the obvious way and by a module V over a Γ-ring R we mean an R-module in the usual sense which is also a Γ-module in such a way that $(\gamma r)v = \gamma(rv) = r(\gamma v)$ for $\gamma \in \Gamma$, $r \in R$, $v \in V$. In particular R itself is an R-module. If V is an R-module we write $\mathcal{E}_\Gamma(V)$ for the ring of Γ-endomorphisms of V and $\mathcal{E}_R(V)$ for the subring of $\mathcal{E}_\Gamma(V)$ consisting of Γ-endomorphisms which are also R-endomorphisms. We note that $\mathcal{E}_\Gamma(V)$ is an R-module and $\mathcal{E}_R(V)$ a submodule. By a *Lie ring* we mean as usual an abelian group $\mathfrak{L}$ together with a pairing $(x, y) \to [x, y]$ of $\mathfrak{L} \times \mathfrak{L}$ into $\mathfrak{L}$ such that $[x, x] = 0$ for all $x \in \mathfrak{L}$ (and hence $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{L}$) and (Jacobi identity) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z, \in \mathfrak{L}$. If R is a Γ-ring and V is an R-module then $\mathcal{E}_\Gamma(V)$ is a Lie ring under the commutator pairing $(x, y) \to [x, y] = xy - yx$ and $\mathcal{E}_R(V)$ is a sub-Lie ring. Moreover we note that the commutator pairing in $\mathcal{E}_\Gamma(V)$ is bilinear over Γ. By a *derivation* of the Γ-ring R we mean a Γ-endomorphism X of R satisfying $X(rs) = (Xr)s + r(Xs)$. It is easily cehcked that the set $\mathfrak{D}(R)$ of derivations of R is a sub R-module of $\mathcal{E}_\Gamma(R)$ and also a sub-Lie ring of $\mathcal{E}_\Gamma(R)$ and that for $X, Y \in \mathfrak{D}(R)$ and $r \in R$ we have

$$[rX, Y] = r[X, Y] - (Yr)X \tag{1}$$

Definition. Let R be a Γ-ring and $\mathfrak{L}$ a Lie ring whose underlying abelian group has the structure of an R-module and suppose that the pairing $(x, y) \to [x, y]$ of $\mathfrak{L} \times \mathfrak{L}$ into $\mathfrak{L}$ is bilinear over Γ. We shall call $\mathfrak{L}$ a *Lie d-ring over R* if there is given as additional structure a map of $\mathfrak{L}$ into $\mathfrak{D}(R)$ which is a homomorphism of the Lie ring and R-module structures such that (denoting the image of $X \in \mathfrak{L}$ on $r \in R$ by Xr) the relation (1) above holds for $X, Y \in \mathfrak{L}$ and $r \in R$.

If the map of $\mathfrak{L}$ into $\mathfrak{D}(R)$ defining the Lie d-ring structure of $\mathfrak{L}$ is the zero map then $\mathfrak{L}$ is called *d-trivial.* In this case the pairing of $\mathfrak{L} \times \mathfrak{L}$ into $\mathfrak{L}$ is bilinear over R.

Examples. (a) R any Γ-ring, $\mathfrak{L} = \mathfrak{D}(R)$ and $\mathfrak{L} \to \mathfrak{D}(R)$ the identity map.

(b) Γ = real numbers, R = differentiable real valued functions on some differentiable manifold M, $\mathfrak{L}$ = differentiable vector fields on M. Since the natural map of $\mathfrak{L}$ into $\mathfrak{D}(R)$ is an isomorphism onto, this is in fact a special case of (a).

(c) R a field, Γ any subring of R and $\mathfrak{L}$ a Lie algebra over R. Then $\mathfrak{L}$ is a d-trivial d-ring over R.

We next define a concept meant to generalize the notion of a linear representation in example (c) above.

DEFINITION. Let R be a Γ-ring, $\mathfrak{L}$ a Lie d-ring over R, and let V be an R-module. We shall call V an *$\mathfrak{L}$-module* if there is given as additional structure a map of $\mathfrak{L}$ into $\mathcal{E}_\Gamma(V)$ which is a homomorphism of the Lie ring and R-module structures such that (denoting the image of $X \in \mathfrak{L}$ on $v \in V$ by Xv) the relation

$$(2) \qquad X(rv) = (Xr)v + r(Xv)$$

holds for $X \in \mathfrak{L}$, $r \in R$, and $v \in V$. We consider R itself as an $\mathfrak{L}$-module called the *basic $\mathfrak{L}$-module* by means of the map $\mathfrak{L} \to \mathfrak{D}(R) \subseteq \mathcal{E}_\Gamma(R)$ which defines the d-ring structure of $\mathfrak{L}$ (condition (2) is just the condition for a derivation in this case).

We note that the direct sum of two $\mathfrak{L}$-modules is in a natural way an $\mathfrak{L}$-module.

If V_1, V_2, V_3 are three $\mathfrak{L}$-modules a map $V_1 \times V_2$ into V_3, say $(v_1, v_2) \to v_1v_2$, will be called an *$\mathfrak{L}$-pairing* if it is a pairing in the sense of R-modules (i.e., bilinear over R *and* Γ) and in addition satisfies

$$(3) \qquad X(v_1v_2) = (Xv_1)v_2 + v_1(Xv_2)$$

for $X \in \mathfrak{L}$, $v_i \in V_i$.

We note that (2) says that the map $(r, v) \to rv$ is an $\mathfrak{L}$-pairing of $R \times V$ into V where R is the basic $\mathfrak{L}$-module and V any $\mathfrak{L}$-module. An $\mathfrak{L}$-module V will be called an *$\mathfrak{L}$-ring* if there is given as additional structure an associative and commutative $\mathfrak{L}$-pairing of $V \times V$ into V. This amounts to saying that V is a commutative ring over R and that the mapping of $\mathfrak{L}$ into $\mathcal{E}_\Gamma(V)$ has its range in $\mathfrak{D}(V)$, the set of derivations of V. In particular then the basic $\mathfrak{L}$-module is always an $\mathfrak{L}$-ring.

2. Cochains of a Lie d-ring with coefficients in a module. Let R be a Γ-ring, $\mathfrak{L}$ a Lie d-ring over R and V an $\mathfrak{L}$-module. For each integer $p \geqq 0$ we define an R-module $C^p(\mathfrak{L}, V)$ called the R-module of p-cochains of $\mathfrak{L}$ with coefficients in V as follows: for $p = 0$ we put $C^p(\mathfrak{L}, V) = V$ and for $p > 0$ we define $C^p(\mathfrak{L}, V)$ to be the set of maps of $\mathfrak{L}^p$ into V which are R and Γ linear in each argument and skew-symmetric in each pair of arguments. The module operations in $C^p(\mathfrak{L}, V)$ are "pointwise". We define the module of cochains of $\mathfrak{L}$ with coefficients in V to be the weak direct sum $C^*(\mathfrak{L}, V) = \bigoplus_{p \geqq 0} C^p(\mathfrak{L}, V)$. We define an R-module automorphism $c \to \bar{c}$ of $C^*(\mathfrak{L}, V)$ with itself by $\bar{c} = (-1)^p c$ for $c \in C^p(\mathfrak{L}, V)$.

EXAMPLES. (a) In example (b) of §1 if we take $V = R$ the basic $\mathfrak{L}$-module, then a p-cochain is a function c which assigns to each p-tuple of differentiable vector fields $X_1, \cdots, X_p$ on M a differentiable function $c(X_1, \cdots, X_p)$ on M, the assignment being p-linear over functions and alternating. If $x \to c_x$ is a differentiable p-form on M then $(X_1, \cdots, X_p) \to c(X_1, \cdots, X_p)$ where $c(X_1, \cdots, X_p)(x) = c_x((X_1)_x, \cdots, (X_p)_x)$ is clearly such a function. That conversely every such function arises from a unique p-form is a well-known and easily proved lemma.[3] Thus in this case the p-cochains are just the p-forms, i.e., the p-cochains of the de Rham complex.

[3] *Added in proof.* For a proof see Proposition 3.4 of A. Frölicher and A. Nijenhuis, *Theory of vector-valued forms.* I, Indag. Math. vol. 18 (1956) p. 338.

b) In example (c) of §1 $C^*(\mathfrak{L}, V)$ coincides with the definition given by Chevalley and Eilenberg in [**1**].

2*. Let $(v_1, v_2) \to v_1 v_2$ be an $\mathfrak{L}$-pairing of two $\mathfrak{L}$-modules V_1 and V_2 into a third V_3. We define an associated map $(c_1, c_2) \to c_1 \wedge c_2$ of $C^*(\mathfrak{L}, V_1) \times C^*(\mathfrak{L}, V_2) \to C^*(\mathfrak{L}, V_3)$ which is an R-pairing (i.e., bilinear over R and Γ) by defining an R-pairing of $C^p(\mathfrak{L}, V_1) \times C^q(\mathfrak{L}, V_2)$ into $C^{p+q}(\mathfrak{L}, V_3)$ and then extending in the obvious way. We put

$$c_1 \wedge c_2(X_1, \cdots, X_q) = c_1(c_2(X_1, \cdots, X_q)), \qquad p = 0, \tag{3a}$$

$$c_1 \wedge c_2(X_1, \cdots, X_p) = c_1(X_1, \cdots, X_p)c_2, \qquad q = 0, \tag{3b}$$

$$c_1 \wedge c_2(X_1, \cdots, X_{p+q}) = \frac{1}{p!q!} \sum_{\omega \varepsilon S_{p+q}} \epsilon(\omega) c_1(X_{\omega(1)}, \cdots, X_{\omega(p)}) c_2(X_{\omega(p+1)}, \cdots, X_{\omega(p+q)}) \tag{3c}$$

for $p, q \neq 0$ where S_{p+q} is the group of permutations of $(1, \cdots, p+q)$ and $\epsilon(\omega)$ is the parity of ω. Now suppose that $V_2 = V_3$ and that we are given in addition a commutative $\mathfrak{L}$-pairing of $V_1 \times V_1$ into V_1 such that $(vv')v'' = v(v'v'')$ for (v, v', v'') in $V_1 \times V_1 \times V_2$. Then a straightforward computation shows that $(c \wedge c') \wedge c'' = c \wedge (c' \wedge c'')$ for $(c, c', c'') \varepsilon C^*(\mathfrak{L}, V_1) \times C^*(\mathfrak{L}, V_1) \times C^*(\mathfrak{L}, V_2)$ and $c_p \wedge c_q = (-1)_{pq} c_q \wedge c_p$ for $c_i \varepsilon C^i(\mathfrak{L}, V_1)$. Thus if V is an $\mathfrak{L}$-ring we can take $V = V_1 = V_2 = V_3$ in the above and we see that $C^*(\mathfrak{L}, V)$ becomes an anticommutative graded ring over R and Γ. In particular if R is the basic $\mathfrak{L}$-module then $C^*(\mathfrak{L}, R)$ is an anticommutative graded ring. Moreover if V is any $\mathfrak{L}$-module then the pairing $(r, v) \to rv$ of $R \times V$ into V satisfies the above hypotheses $(R = V_1, V = V_2 = V_3)$ and we see that $C^*(\mathfrak{L}, V)$ is a module over $C^*(\mathfrak{L}, R)$.

3. **Interior products.** Let $\mathfrak{L}$ be a Lie d-ring over the Γ-ring R and let V be an $\mathfrak{L}$-module. For each $X \varepsilon \mathfrak{L}$ we define an endomorphism of degree -1 of the graded R-module $C^*(\mathfrak{L}, V)$ called the *interior product* by X and denoted by i_X as follows:

$$i_X c = 0 \quad \text{for} \quad c \varepsilon C^0(\mathfrak{L}, V) = V,$$

$$i_X c(X_1, \cdots, X_{p-1}) = c(X, X_1, \cdots, X_{p-1}), \quad c \varepsilon C^p(\mathfrak{L}, V), p > 0.$$

We leave to the reader the easy verification that $X \to i_X$ is a R-module homomorphism of $\mathfrak{L}$ into $\mathcal{E}_R(C^*(\mathfrak{L}, V))$. It is not a Lie ring homomorphism for in fact we have the following anti-commutator relation

$$i_X i_Y + i_Y i_X = 0.$$

1*. Assuming as in 2* that we have an $\mathfrak{L}$-pairing $V_1 \times V_2 \to V_3$ we ask how i_X behaves with respect to the associated pairing

$$C^*(\mathfrak{L}, V_1) \times C^*(\mathfrak{L}, V_2) \to C^*(\mathfrak{L}, V_3).$$

A straightforward computation shows that

$$i_X(c_1 \wedge c_2) = (i_X c_1) \wedge c_2 + \bar{c}_1 \wedge (i_X c_2) \tag{7}$$

where $c \to \bar{c}$ is as defined in §1. If V is an $\mathfrak{L}$-ring then (7) says that i_X is an anti-derivation of the graded ring $C^*(\mathfrak{L}, V)$.

4. The Lie derivative. Again $\mathfrak{L}$ is a Lie d-ring over R and V an $\mathfrak{L}$-module. We now define for each $X \in \mathfrak{L}$ a Γ-endomorphism ∂_X of $C^*(\mathfrak{L}, V)$, homogeneous to degree zero, called the Lie derivative with respect to X, by

$$\partial_X c = Xc \quad \text{if} \quad c \in C^0(\mathfrak{L}, V) = V, \tag{8a}$$

$$\begin{aligned} &\partial_X c(X_1 \cdots X_p) \\ &= Xc(X_1, \cdots, X_p) - \sum_{i=1}^{p} c(X_1, \cdots, X_{i-1}, [X, X_i], X_{i+1}, \cdots, X_p) \\ &= Xc(X_1, \cdots, X_p) + \sum_{i=1}^{p} (-1)^i c([X, X_i], X_1, \cdots, \hat{X}_i \cdots, X_p) \\ &\qquad\qquad \text{if} \quad c \in C^p(\mathfrak{L}, V), p > 0. \end{aligned} \tag{8b}$$

It is clear that $\partial_X c$ as defined is skew-symmetric in each pair of arguments and Γ-linear in each argument. To show that $\partial_X c$ is R-linear in each argument we note that since

$$[X, rX_i] = r[X, X_i] + (Xr)X_i$$

it follows that

$$\begin{aligned} \partial_X c(X_1, \cdots, rX_i, \cdots, X_p) &= X(rc(X_1, \cdots, X_p)) \\ &\quad - \sum_{i=1}^{p} rc(X_1, \cdots, [X, X_i], \cdots, X_p) - (Xr)c(X_1, \cdots, X_p) \\ &= rXc(X_1, \cdots, X_p) - r\sum_{i=1}^{p} c(X_1, \cdots, [X, X_i], \cdots, X_p) \\ &= r\,\partial_X c(X_1, \cdots, X_p). \end{aligned}$$

It is now clear that ∂_X is a Γ-endomorphism of each $C^p(\mathfrak{L}, V)$ and so extends uniquely to a Γ-endomorphism of $C^*(\mathfrak{L}, V)$. Unlike i_X however, ∂_X is not an R-endomorphism in general for in fact it is easily seen that $\partial_X(rc) = r\partial_X c + (Xr)c$. The map $X \to \partial_X$ of $\mathfrak{L}$ into $\mathcal{E}_\Gamma(C^*(\mathfrak{L}, V))$ is obviously a Γ-module endomorphism and using the Jacobi identity we get easily that

$$\partial_{[X,Y]} = [\partial_X, \partial_Y] \tag{9}$$

which is to say that it is also a Lie ring homomorphism. Finally the commutator of a Lie derivative ∂_Y and an inner product i_X is easily computed and turns out to be another interior product

$$[\partial_Y, i_X] = i_{[X,Y]}. \tag{10}$$

4*. Once again assuming that we have an $\mathfrak{L}$-pairing $V_1 \times V_2 \to V_3$ we now ask how ∂_X behaves relative to the associated pairing of cochain modules. Another direct computation shows

$$(11) \qquad \partial_X(c_1 \wedge c_2) = (\partial_X c_1) \wedge c_2 + c_1 \wedge \partial_X c_2 .$$

In particular if V is a Lie ring then each ∂_X is a derivation of the graded ring $C^*(\mathfrak{L}, V)$.

5. Invariant cochains. If $\mathfrak{L}$ is a Lie d-ring over a Γ-ring R and V is and $\mathfrak{L}$-module we define $C_I^*(\mathfrak{L}, V)$, the set of *invariant cochains* of $\mathfrak{L}$ with coefficients in V by $C_I^*(\mathfrak{L}, V) = \{c \in C(\mathfrak{L}, V) \mid \partial_X c = 0 \text{ for all } X \in \mathfrak{L}\}$. Since each ∂_X is a Γ-endomorphism of $C^*(\mathfrak{L}, V)$ it follows that $C_I^*(\mathfrak{L}, V)$ is a Γ-submodule of $C^*(\mathfrak{L}, V)$ (but not in general and R-submodule) and since each ∂_X is homogeneous of degree zero it follows that $C_I^*(\mathfrak{L}, V)$ is the direct sum of the Γ-submodules $C_I^p(\mathfrak{L}, V) = C^p(\mathfrak{L}, V) \cap C_I^*(\mathfrak{L}, V)$ of invariant p-cochains.

5*. Assuming again an $\mathfrak{L}$-pairing of $V_1 \times V_2$ into V_3 it follows from (11) that if $c_i \in C_I^*(\mathfrak{L}, V_i)$, $i = 1, 2$, then $c_1 \wedge c_2 \in C_I^*(\mathfrak{L}, V_3)$. In particular if V is an $\mathfrak{L}$-ring then $C_I^*(\mathfrak{L}, V)$ is a subring of $C^*(\mathfrak{L}, V)$.

6. The differential or coboundary operator. If $\mathfrak{L}$ is a Lie d-ring over a Γ-ring R then for each $\mathfrak{L}$-module V we define $d \in \mathcal{E}_\Gamma(C^*(\mathfrak{L}, V))$, homogeneous of degree $+1$ as follows:

$$(12a) \qquad dv(X) = Xv, \qquad v \in C^0(\mathfrak{L}, V) = V,$$

$$(12b) \qquad \begin{aligned} dc(X_1, \cdots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} X_i c(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1}) \\ &+ \sum_{i<j} (-1)^{i+j} c([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{p+1}) \\ &\qquad \text{if } c \in C^p(\mathfrak{L}, V), p > 0. \end{aligned}$$

Using (8b) we see (12b) can be rewritten

$$(13) \qquad \begin{aligned} dc(X_1, \cdots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} (\partial_{X_i} c)(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1}) \\ &+ \sum_{i<j} (-1)^{i+j+1} c([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{p+1}) \end{aligned}$$

or if $1 + 1$ has an inverse, $1/2$, in Γ we can write

$$(14) \qquad \begin{aligned} dc(X_1 \cdots X_{p+1}) = \tfrac{1}{2} \sum_{i=1}^{p+1} (-1)^{i+1} \{ &X_i C(X_1 \cdots \hat{X}_i \cdots X_{p+1}) \\ &+ (\partial_{X_i} c)(X_1 \cdots \hat{X}_i \cdots X_{p+1}) \}. \end{aligned}$$

It is clear from any of these forms of the definitions that dc is anti-symmetric in each pair of arguments and Γ-linear in each argument. That dc is R-linear in each argument is not so obvious but of course it suffices to prove R-linearity in the first argument. Now using (12b) and (1)

$$
\begin{aligned}
dc(rX_1, X_2, \cdots, X_{p+1}) &= r \sum_{i=1}^{p+1} (-1)^{i+1} X_i c(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1}) \\
&+ \sum_{i=2}^{p+1} X_i(r) c(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1}) \\
&+ r \sum_{i<j} (-1)^{i+j} c([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{p+1}) \\
&+ \sum_{1<j} (-1)^{1+j} c(-X_j(r) X_1, X_2, \cdots, \hat{X}_j, \cdots, X_{p+1}).
\end{aligned}
$$

The second and fourth terms cancel and the remaining give $r\, dc(X_1, \cdots, X_{p+1})$. It is clear that $c \to dc$ is a Γ-homomorphism of each $C^p(\mathfrak{L}, V)$ into $C^{p+1}(\mathfrak{L}, V)$ and hence extends uniquely to a Γ-endomorphism of degree $+1$ of $C^*(\mathfrak{L}, V)$.

Using (13) and the commutation rule (10) we get by direct computation the anti-commutation rule

(15) $$\partial_X = di_X + i_X d.$$

Now just as in [**1**, p. 116] we show in turn that $d\partial_X = \partial_X d$ and that

(16) $$d \circ d = 0.$$

Using (15) and (16) we can find the common value of $d\partial_X$ and $\partial_X d$, namely

(17) $$d\partial_X = \partial_X d = di_X d.$$

However the real importance of (16) of course is that it says that $(C^*(\mathfrak{L}, V), d)$ is a graded cochain complex with a differential of degree $+1$. We will use all the standard notation and terminology that this implies. We note that since d is a Γ-endomorphism the cocycle group $Z^*(\mathfrak{L}, V) = \bigoplus_p Z^p(\mathfrak{L}, V)$, the coboundary group $B^*(\mathfrak{L}, V) = \bigoplus_p B^p(\mathfrak{L}, V)$ and the cohomology group $H^*(\mathfrak{L}, V) = \bigoplus_p H^p(\mathfrak{L}, V)$ are all Γ-modules.

It follows from (17) that the invariant cochains $C_I^*(\mathfrak{L}, V)$ of an $\mathfrak{L}$-module V form a sub-cochain complex of $C^*(\mathfrak{L}, V)$ giving rise therefore to the notions of invariant cocycles $Z_I^*(\mathfrak{L}, V) = Z^*(\mathfrak{L}, V) \cap C_I^*(\mathfrak{L}, V)$, invariant coboundaries $B_I^*(\mathfrak{L}, V) = d\, C_I^*(\mathfrak{L}, V)$ (*not* the possibly larger $B^*(\mathfrak{L}, V) \cap C_I^*(\mathfrak{L}, V)$) and invariant cohomology group $H_I^*(\mathfrak{L}, V) = Z_I^*(\mathfrak{L}, V)/B_I^*(\mathfrak{L}, V)$ all of which are again graded Γ-modules.

EXAMPLES. (a) In example (a) of §2 d is the usual exterior derivative. For a proof see [**2**].

(b) In example (b) of §2 d is defined precisely as in [**1**].

6*. Assuming once again an $\mathfrak{L}$-pairing of two $\mathfrak{L}$-modules V_1 and V_2 into a third V_3 we ask for the relation of d to the related pairing $C^*(\mathfrak{L}, V_1) \times C^*(\mathfrak{L}, V_2) \to C^*(\mathfrak{L}, V_3)$. Once again straightforward computation leads to the desired formula, namely

(18) $$d(c_1 \wedge c_2) = (dc_1) \wedge c_2 + \bar{c}_1 \wedge (dc_2).$$

This allows us to compute how much d misses being an R-endomorphism, for taking V_1 to be the basic $\mathfrak{L}$-module R and $V_2 = V$ an arbitrary $\mathfrak{L}$-module we get

$$d(rc) = r\,dc + dr \wedge c. \tag{19}$$

We can now use (19) and (15) and the fact that $i_{rX} = ri_X$ to see how much $X \to \partial_X$ misses being an R-homomorphism. The result is

$$\partial_{rX} = r\,\partial_X + dr \wedge i_X \tag{20}$$

where $(dr \wedge i_X)c = dr \wedge (i_X c)$.

However the most important fact about (18) is that it says that in case V is an $\mathfrak{L}$-ring d is an antiderivation of the anti-commutative ring $C^*(\mathfrak{L}, V)$ and hence that $B^*(\mathfrak{L}, V)$ is an ideal in the subring $Z^*(\mathfrak{L}, V)$ so that $H^*(\mathfrak{L}, V)$ has the structure of an anti-commutative graded ring over Γ. In the same way $H^*_I(\mathfrak{L}, V)$ also has such a structure.

7. Cohomology of Lie *d*-rings and of Γ-rings. If R is any Γ-ring then for each Lie d-ring $\mathfrak{L}$ over R we define $H^*(\mathfrak{L})$, the cohomology group of $\mathfrak{L}$, to be the graded Γ-module $H^*(\mathfrak{L}, R)$ where R is the basic $\mathfrak{L}$-module. In case Γ contains as a subring the rational numbers we consider $H^*(\mathfrak{L})$ as an anti-commutative graded ring as noted in §6*. Similarly we define the invariant cohomology of $\mathfrak{L}$, $H^*_I(\mathfrak{L})$ by $H^*_I(\mathfrak{L}) = H^*_I(\mathfrak{L}, R)$.

If R is any Γ-ring we define the cohomology and invariant cohomology of R by

$$H^*(R) = H^*(\mathfrak{D}(R)) = H^*(\mathfrak{D}(R), R),$$
$$H^*_I(R) = H^*_I(\mathfrak{D}(R)) = H^*_I(\mathfrak{D}(R), R),$$

where as usual $\mathfrak{D}(R)$ is the set of derivations of R considered as a Lie d-ring. Once again if Γ includes the rational numbers as a subring then $H^*(R)$ and $H^*_I(R)$ have the structure of an anti-commutative graded ring. Since all constructions have been canonical it is clear that every Γ-isomorphism of a Γ-ring R onto a Γ-ring R' induces an isomorphism of $H^*(R)$ onto $H^*(R')$ in a natural way. In other words the map $R \to H^*(R)$ is a functor from the category of Γ-rings with Γ-isomorphisms as maps to the category of graded Γ-modules (anti-commutative graded rings over Γ in case $Q \subseteq \Gamma$) with homogeneous Γ-isomorphisms. Moreover as follows from the remarks made in Examples of §§2 and 6 if Γ is the field of real numbers and R the ring of differentiable real valued functions on a differentiable manifold M then $H^*(R)$ is by de Rham's theorem the real cohomology algebra of M.

Bibliography

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