A Disorienting Look at Euler's Theorem on the Axis of a Rotation

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1. INTRODUCTION. A rotation in two dimensions (or other even dimensions) does not in general leave any direction fixed, and even in three dimensions it is not immediately obvious that the composition of rotations about distinct axes is equivalent to a rotation about a single axis. However, in 1775–1776, Leonhard Euler [**8**] published a remarkable result stating that in three dimensions every rotation of a sphere about its center has an axis, and providing a geometric construction for finding it.

In modern terms, we formulate Euler's result in terms of rotation matrices as follows.

Euler's Theorem on the Axis of a Three-Dimensional Rotation. *If* $\mathbf{R}$ *is a* 3×3 *orthogonal matrix (*$\mathbf{R^T R} = \mathbf{RR^T} = \mathbf{I}$*) and* $\mathbf{R}$ *is proper (*det $\mathbf{R} = +1$*), then there is a nonzero vector* $\mathbf{v}$ *satisfying* $\mathbf{Rv} = \mathbf{v}$.

This important fact has a myriad of applications in pure and applied mathematics, and as a result there are many known proofs. It is so well known that the general concept of a rotation is often confused with rotation about an axis.

In the next section, we offer a slightly different formulation, assuming only orthogonality, but not necessarily orientation preservation. We give an elementary and constructive proof that appears to be new that there is either a fixed vector or else a "reversed" vector, i.e., one satisfying $\mathbf{Rv} = -\mathbf{v}$. In the spirit of the recent tercentenary of Euler's birth, following our proof it seems appropriate to survey other proofs of this famous theorem. We begin with Euler's own proof and provide an English translation from the original Latin. Euler's construction relies on implicit assumptions of orientation preservation and genericity, and leaves confirmation of his characterization of the fixed axis to the reader. Our current tastes prefer such matters to be spelled out, and we do so in Section 4. There, we again classify general distance preserving transformations, this time using Euler's spherical geometry in modern dress instead of linear algebra. We note that some constructions present in Euler's original paper correspond to those appearing in our proof with matrices. In the final section, we survey several other proofs.

2. EULER'S THEOREM FOR ORTHOGONAL MATRICES: A CONSTRUCTIVE LINEAR ALGEBRA PROOF. We will see below that, in Euler's original paper, preservation of orientation is assumed implicitly. In this section we omit any assumption that our map $\mathbf{R}$ is either proper or improper (det $\mathbf{R} = -1$), and infer it from the conclusion.

Euler's Theorem for 3 × 3 Orthogonal Matrices. *If* $\mathbf{R}$ *is a* 3×3 *orthogonal matrix, then there is a nonzero vector* $\mathbf{v}$ *satisfying* $\mathbf{Rv} = \mathbf{v}$*, or a nonzero vector* $\mathbf{v}$ *satisfying* $\mathbf{Rv} = -\mathbf{v}$.

Remark. The proof will find all such vectors constructively, and will therefore identify the dimension of the subspace of reversed vectors $\mathbf{v}$ satisfying $\mathbf{R}\mathbf{v} = -\mathbf{v}$. $\mathbf{R}$ is proper or improper according to whether this dimension is even or odd.

Proof. We begin the proof by observing that if $\mathbf{A} := \frac{1}{2}(\mathbf{R} - \mathbf{R^T})$ is the skew-symmetric part of $\mathbf{R}$, then $\mathbf{R}$ commutes with $\mathbf{A}$:

$$\mathbf{RA} = \mathbf{AR}. \tag{1}$$

To see this, we use the fact that $\mathbf{R}$ commutes with itself and with its inverse $\mathbf{R}^{-1} = \mathbf{R^T}$, so it commutes with $\mathbf{R} - \mathbf{R^T}$ and hence with $\mathbf{A}$. As a consequence, if $\mathbf{v}$ is in the kernel of $\mathbf{A}$, $\mathbf{A}(\mathbf{R}\mathbf{v}) = \mathbf{R}(\mathbf{A}\mathbf{v}) = \mathbf{0}$ shows that $\mathbf{R}\mathbf{v}$ is also in the kernel of $\mathbf{A}$.

Our proof proceeds by treating two cases: a generic case, $\mathbf{A} \neq \mathbf{0}$, and a nongeneric case, $\mathbf{A} = \mathbf{0}$. ∎

1. The Generic Case. Note that $\mathbf{A}$ takes the form:

$$\mathbf{A} = \begin{pmatrix} 0 & a_{12} & -a_{31} \\ -a_{12} & 0 & a_{23} \\ a_{31} & -a_{23} & 0 \end{pmatrix}.$$

If we let $\mathbf{v} := (a_{23} \; a_{31} \; a_{12})^T$ then $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{A}\mathbf{v} = \mathbf{0}$ is an identity, i.e., $\mathbf{A}$ has nontrivial kernel. The form of $\mathbf{A}$ guarantees that, if it is nonzero, its rank is at least 2. Since $\mathbf{A}$ has nonzero kernel, its rank must be exactly 2 and therefore $\mathbf{v}$ spans its kernel. Since $||\mathbf{R}\mathbf{v}|| = ||\mathbf{v}||$, this shows $\mathbf{R}\mathbf{v} = \pm\mathbf{v}$.

Remark. In the generic case, there can be no vectors independent of $\mathbf{v}$ satisfying $\mathbf{R}\mathbf{v} = \pm\mathbf{v}$. For $\mathbf{R}\mathbf{v} = \pm\mathbf{v}$ implies that $(\mathbf{R} - \mathbf{R}^{-1})\mathbf{v} = \mathbf{0}$, so $\mathbf{v}$ is in the (one-dimensional) kernel of $\mathbf{A}$. Since $\mathbf{R}$ preserves orthogonality, the orthogonal complement of $\mathbf{v}$, $\mathbf{v}^\perp$, is an invariant two-dimensional subspace on which $\mathbf{R}$ acts orthogonally. We can find its matrix with respect to an orthonormal basis for $\mathbf{v}^\perp$ by observing that the image of the first basis vector is some arbitrary unit vector in $\mathbf{v}^\perp$, $\binom{a}{b}$, and the image of the second is one of the two unit vectors orthogonal to the image of the first, $\pm \left(\begin{smallmatrix} -b \\ a \end{smallmatrix}\right)$. If we choose the minus sign, the resulting $\mathbf{R}$ is $\left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$ which is a reflection across a fixed vector. In this case, we see that $\mathbf{R}$ is nongeneric, because it is symmetric with respect to an orthonormal basis, or alternatively, since $\mathbf{R}^2 = \mathbf{I}$ so $\mathbf{R}^T = \mathbf{R}^{-1} = \mathbf{R}$. Since we have assumed $\mathbf{R}$ is generic, $\mathbf{R}$ must act as a proper rotation of $\mathbf{v}^\perp$, and we can see that $\mathbf{R}$ preserves the orientation of a right-handed frame if $\mathbf{v}$ is fixed, and reverses its orientation if $\mathbf{v}$ is reversed. We will say more about orientation preservation after treating the nongeneric case. It is possible to show that $\mathbf{A}$ has nontrivial kernel in any odd dimension n, by appealing to $\det \mathbf{A} = \det \mathbf{A}^T = \det -\mathbf{A} = (-1)^n \det \mathbf{A}$.

The above case is generic because the orthogonal matrices with $\mathbf{A} = \mathbf{0}$, i.e., the symmetric ones, have measure zero, corresponding to angles of rotation 0 and π. Still they are important, and we have a similar result for them.

2. The Nongeneric Case. If $\mathbf{R}$ is an $n \times n$ matrix, then we can write any vector $\mathbf{x} \in \mathbb{R}^n$ in the form $\mathbf{x} = \mathbf{x}_+ + \mathbf{x}_-$, where $\mathbf{x}_+ = \frac{1}{2}(\mathbf{x} + \mathbf{R}\mathbf{x})$ and $\mathbf{x}_- = \frac{1}{2}(\mathbf{x} - \mathbf{R}\mathbf{x})$. When $\mathbf{A} = \mathbf{0}$ and $\mathbf{R} = \mathbf{R}^T = \mathbf{R}^{-1}$, we have $\mathbf{R}^2 = \mathbf{I}$, and in this case $\mathbf{R}\mathbf{x}_+ = +\mathbf{x}_+$, $\mathbf{R}\mathbf{x}_- = -\mathbf{x}_-$. Moreover, for any $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x}_+^T\mathbf{y}_- = 0$. In other words, we have an orthogonal decomposition $\mathbb{R}^n = \mathbf{V}^+ \oplus \mathbf{V}^-$ into the fixed-point set $\mathbf{V}^+$ (the $+1$ eigenspace) of $\mathbf{R}$

and the space $\mathbf{V}^-$ of reversed vectors (the -1 eigenspace) of $\mathbf{R}$, and the operators $\mathbf{P}_+ = \frac{1}{2}(\mathbf{I} + \mathbf{R})$ and $\mathbf{P}_- = \frac{1}{2}(\mathbf{I} - \mathbf{R})$ are the projections ($\mathbf{P}_{\pm}^2 = \mathbf{P}_{\pm}$) onto these spaces. From the point of view of operator notation, this is a consequence of $\mathbf{R}^2 = \mathbf{I}$, which implies $\mathbf{R}^2 - \mathbf{I} = (\mathbf{R} + \mathbf{I})(\mathbf{R} - \mathbf{I}) = (\mathbf{R} - \mathbf{I})(\mathbf{R} + \mathbf{I}) = \mathbf{0}$.

For $n = 3$, we may now classify symmetric orthogonal 3×3 matrices into four cases according to the complementary dimensions d_+ and d_- of the fixed and reversed subspaces. As in the generic case, we note that $\mathbf{R}$ is proper or improper according to whether d_- is even or odd. If $d_+ = 0$ and $d_- = 3$ then $\mathbf{R}$ is the antipodal map (improper). If $d_+ = 1$ and $d_- = 2$ then $\mathbf{R}$ is a half-turn rotation about some axis (proper). If $d_+ = 2$ and $d_- = 1$ then $\mathbf{R}$ is a reflection in some plane (improper). And finally, if $d_+ = 3$ and $d_- = 0$ then $\mathbf{R}$ is the identity map (proper).

It is possible to formulate the following converse to the classification of nongeneric 3×3 orthogonal matrices as well. If $\mathbf{R}$ is an orthogonal 3×3 matrix that fixes or reverses at least two independent vectors, then $\mathbf{R}$ is symmetric. The proof is elementary and left to the reader.

Including the proper and improper generic cases, we have in total six cases, which are characterized by the dimensions of their fixed and reversed subspaces.

Examples.

1. If
$$\mathbf{R} = \mathbf{R}_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$\mathbf{A} = \begin{pmatrix} 0 & \sin\theta & 0 \\ -\sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ \sin\theta \end{pmatrix}.$$

With $\theta \neq 0, \pi$, we are in the generic case, the z-axis is fixed, and $\mathbf{R}$ is proper.

2. If
$$\mathbf{R} = \mathbf{R}_G = \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{pmatrix},$$

then

$$\mathbf{A} = \begin{pmatrix} 0 & 1/2 & -1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1/2 & 0 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

In this case $\mathbf{v}$ is fixed and $\mathbf{R}$ is proper.

3. If
$$\mathbf{R} = \mathbf{R}_N = \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix},$$

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then $\mathbf{A} = \mathbf{0}$, so $\mathbf{R}$ is nongeneric. Thus we form

$$\mathbf{R} + \mathbf{I} = \begin{pmatrix} 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{pmatrix} \text{ and } \mathbf{R} - \mathbf{I} = \begin{pmatrix} -4/3 & 2/3 & 2/3 \\ 2/3 & -4/3 & 2/3 \\ 2/3 & 2/3 & -4/3 \end{pmatrix}.$$

We find that $\mathbf{x}_+ = (2/3 \ 2/3 \ 2/3)^T$ is fixed, while $\mathbf{x}_{-,1} = (-4/3 \ 2/3 \ 2/3)^T$ and $\mathbf{x}_{-,2} = (2/3 \ -4/3 \ 2/3)^T$ span the reversed subspace.

4. If

$$\mathbf{R} = \mathbf{R}_0 = \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the identity, then $\mathbf{A} = \mathbf{0}$, and

$$\mathbf{R} + \mathbf{I} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Our construction gives $\mathbf{v} = (2 \ 0 \ 0)^T, (0 \ 2 \ 0)^T, (0 \ 0 \ 2)^T$ as fixed vectors.

5. If

$$\mathbf{R} = \mathbf{R}_\pi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is a rotation about the z-axis by an angle π, then $\mathbf{A} = \mathbf{0}$, so $\mathbf{R}$ is non-generic. We form

$$\mathbf{R} + \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } \mathbf{R} - \mathbf{I} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and we find $\mathbf{x}_+ = (0 \ 0 \ 2)^T$ is fixed while $\mathbf{x}_{-,1} = (-2 \ 0 \ 0)^T$ and $\mathbf{x}_{-,2} = (0 \ -2 \ 0)^T$ are reversed.

Composition with $-\mathbf{I}$ gives a one-to-one correspondence between proper and improper orthogonal transformations. We may use this to obtain improper examples from the proper ones above.

Remarks. Note that no multiplications are required to find an axis in either the generic or nongeneric case. Similarly, the cosine of the angle of rotation about this axis can easily be found by taking the trace of $\mathbf{R}$. The trace is independent of basis, and in a proper orthonormal basis containing the axis, the matrix for $\mathbf{R}$ is given by the matrix $\mathbf{R}_\theta$ in example 1 above, whose trace is clearly $1 + 2\cos\theta$. For comparison, other approaches have been suggested to compute the axis that involve from 1 to 3 cross-products or up to 18 multiplications, followed by more cross-products, normalizations, and projections for up to about 40 multiplications to find the cosine of the angle [**6**, Chapt. 12]. Others have used the Euler-Rodrigues formula, derived with the assumption that an axis exists, to obtain a variety of similar formulas for the axis [**1, 9, 11, 13**], yet none of these claim to be proofs of its existence.

The formulation and proof we have given is independent of whether **R** is proper or improper. We have also shown that the evenness or oddness of the number of independent reversed vectors corresponds to our intuitive notion of a proper or improper transformation preserving or reversing right-handed frames. If we so wish, we can take evenness or oddness of the dimension of the reversed subspace as our definition of a proper or improper orthogonal transformation. As a corollary, proper orthogonal transformations of $\mathbb{R}^3$ always have a nonzero fixed vector.

We could also bring orientation into Euler's theorem by rewriting (1) in the conjugation form $\mathbf{RAR}^{-1} = \mathbf{RAR^T} = \mathbf{A}$, observing that $\mathbf{A}$ acts as a cross-product, $\mathbf{Aw} = \mathbf{v} \times \mathbf{w}$, and invoking the transformation rule for the cross-product under rotation. Any further correspondences would involve developing more advanced definitions of orientation preservation, e.g., determinants. Somewhat in the spirit of Axler [**4, 5**], we prefer to leave this to other treatments.

3. EULER'S GEOMETRIC PROOF: TRANSLATION AND COMMENTARY.
Since we have not found an English translation of Euler's original proof from [**8**] in the literature, we provide one here, followed by a discussion of its meaning. Paragraph numbers 25 and 26 are taken from Euler's original article, as is the above Figure 1 (Euler's Fig. 2). The authors have prepared an interlinear translation of Euler's text (Latin and verbatim English translation on alternate lines) that may be viewed online at `http://vmm.math.uci.edu/euler/interlinear.htm`.
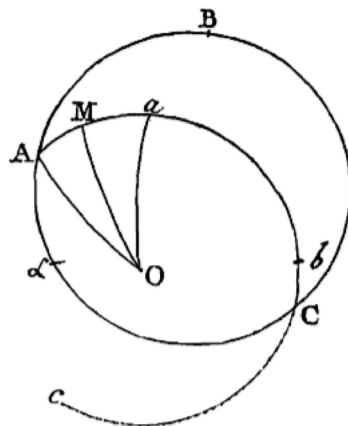

Figure 1. (Euler's Fig. 2)

Theorem. *In whatever way a sphere might be rotated around its own center, a diameter can always be chosen whose direction in the rotated configuration would coincide with the original configuration.*

Demonstration.

25. Let (Fig. 2) circle A, B, C refer to an arbitrary great circle of a sphere in an initial configuration, which after a rotation will attain the configuration a, b, c, in such a way that points A, B, C should be rotated into points a, b, c; also, at the same time let the point A be an intersection point of these two circles. With this setup, it is to be proved that a point O always is given which is related in the same way to circle A, B, C as to circle a, b, c. For this, therefore, it is necessary first that distances OA and Oa be equal

to each other; and next, moreover, it is also necessary that arcs *OA* and *Oa* to those two circles be equally inclined, or equivalently, that the angle *Oab* = the angle *OAB*: and therefore also will be the complements in two right-angles [= the supplementary angles], that is, the angles *OaA* and *OAα*, must be equal. However, since the arcs *Oa* and *OA* are equal, angle *OaA* will also equal angle *OAa*, and therefore *OAa* = *OAα*. From which clearly, if the angle *aAα* is bisected by the arc *OA*, then the point *O* which is sought will be situated somewhere on this arc *AO*. That point, therefore, will be discovered if the arc *aO* is constructed in such a way that angle *AaO* comes out equal to angle *OAa*. In fact, the intersection of these arcs will give the point *O*, through which, if a diameter of the sphere is constructed, its position in the rotated configuration will still be the same as it was in the initial configuration.

26. To define the point *O* more easily, arc *Aa* can be bisected by point *M*, from which is constructed arc *MO* normal to *Aa*. Then certainly arc *AO* can be drawn in such a way that it bisects angle *aAα*; and so the intersection of these arcs will reveal *O*, which is the point sought. Here it is observed, if arc *aα* [*Aα*!!] is taken equal to arc *aA*, *α* will be the point of the sphere which, after the rotation, attains the point *A*, for which reason angle *aAα* ought to be bisected, as opposed to its adjacent angle *aAB*.

In paragraph 25 Euler implicitly uses the fact that the direction and distance of an arbitrary point, relative to a reference point on a reference circle, uniquely characterize its image relative to their images under a rigid rotation **T**, using this to characterize the axis *O* of **T** by equating **T**(*O*) = *O*. If we call the reference point *A*, call its image *a*, and call some other point on the reference circle *α*, then Euler characterizes the axis as the unique point on the bisector of angle *αAa* that makes the angles *OAa* and *AaO* equal. The image of the great circle *l* bisecting angle *αA a* contains both *O* and *a* = **T**(*A*), so this prescription can be viewed as the finding *O* as the intersection of *l* with its own image.

In paragraph 26, Euler presents a second method for constructing an axis in which the bisector of angle *αA a* is intersected with the perpendicular bisector of the arc *Aa*. The midpoint of *Aa* through which it passes is designated *M* in Euler's Figure 2. At the end of this construction Euler notes that when *α* is further specified by **T**(*α*) = *A*, then the lengths of the arcs *aα* and *aA* will be equal. This appears in both printings, although surely he means |*Aα*| = |*aA*|. In this case, the bisector of angle *αA a* agrees with the perpendicular bisector of *α* and *a*. Under a different arrangement of reference points—used by Euler in another context—the angle corresponding to *aAB* is bisected to construct the axis, so the final line emphasizes which of two angles formed by the intersection of a great circle and its image should be bisected. [1]

Euler's characterizations of fixed points are indeed correct for generic, length and orientation preserving motions of a sphere about its center. From a modern point of view, we would require some further detail in a complete proof. In particular, Euler's first construction breaks down when **T** is nongeneric, in which case the intersection point of a circle and its image, *A*, is on the equator, and *a* = **T**(*A*) is its antipodal point. The condition that the angles *OAa* and *OaA* are equal holds for any point *O* on the great circle bisecting the angle between the original circle *ABC* and the image circle *abc*, and cannot be used to locate the axis uniquely.

[1] We are grateful to a reviewer for pointing out the significance of the paper [**7**] that Euler published thirteen years earlier since it illuminates his argument above. In it, Euler finds the instantaneous axis of a rotational motion of a rigid body about its center of gravity, along which the velocity is zero. Since the construction of [**8**] above is based on a somewhat similar characterization of the axis, it is interesting to compare these two papers, and the interested reader can find a link to the original paper [**7**] (in French), our translation, and a discussion of how the constructions of axes in these two papers are related at: `http:math.utah.edu/~palais/E177.html`

Euler starts from the assumption that a fixed point O exists. Based on this assumption, he finds two ways to characterize its location in terms of constructible directions along which one can proceed from certain points, for a given distance, or until one intersects another such great circle. Once this has been accomplished, our modern sensibilities would want a demonstration that these constructions are always well-defined, and that the point that it finds is indeed fixed. Otherwise it is possible to be misled by a compelling figure or self-fulfilling assumption. As we have just pointed out, there are certain cases where the condition specifying the distance of the axis from A is degenerate.

As noted above, the fact that **T** is orientation preserving is no more explicitly invoked in this proof than in that of [**7**]. There, Euler makes use of distance and angle preservation, and proximity to the identity, which combine to identify rotations. In [**8**], only distance and angle preservation are visibly used, so it might seem to apply to improper orthogonal transformations as well, although we know that it cannot. The principle that a distance and angle relative to a point on a circle uniquely determines a location breaks down if we ignore the orientation of the angle, and this is equivalent to ignoring the distinction between proper and improper transformations. Because of this, we might expect Euler to distinguish carefully between angles of opposite orientation, since consistent angle orientation is equivalent to orientation for maps of the sphere. But in fact he equates angle OAa with the oppositely oriented OaA and $OA\alpha$ at various points of his demonstration. We see this as simply an opportunity, or perhaps a suggestion from Euler, that it may be possible to recast his idea in a setting where both orientations are treated simultaneously as we have done below. Without modification, Euler's second construction fails to find the reversed axis of a generic improper orthogonal transformation, since it is no longer on the perpendicular bisector of the arc joining a point and its image (see Figure 4, below). Since the reversed points are on the bisector of the angle $aA\alpha$, the first construction will work if the condition locating them by equating angles is modified appropriately.

Euler's theorem is the first topic discussed in E. T. Whittaker's classic treatise on analytical dynamics [**18**], giving an indication of its central importance in rigid body dynamics. Whittaker considers a rotation of an arbitrary "rigid body" for which "the mutual distances of every pair of specified points is invariable" about an arbitrary point, O, whose position in space is unchanged. He reformulates Euler's theorem using planes in space rather than great circles on a sphere, and provides a proof in the spirit of the Euler's second construction. Others have noted that Whittaker also makes no explicit reference to orientation preservation, though his proof concludes with an argument confirming that the line of intersection of two planes he constructs is indeed fixed. For Whittaker, the condition that a rigid motion is proper may be implicit in the fact that it is path connected to the identity. Since it is nowhere indicated how that fact enters into the proof, by our current standards the proof would be considered incomplete.

4. A GEOMETRIC EULER'S THEOREM FOR DISTANCE-PRESERVING TRANSFORMATIONS.
We now aim to give another proof of the result we proved above using linear algebra, using only the kind of classical spherical geometry arguments that Euler himself used, translated into the modern idiom. We will show that length-preserving transformations of a sphere about its center, both proper and improper, have a fixed or reversed diameter. We will address the generic and nongeneric cases, and in each case confirm rigorously that the diameter we construct is fixed or reversed.

But first, since the theorem of the first section as well as Whittaker's formulation are formulated in the framework of orthogonal linear transformations of $\mathbb{R}^3$, while Euler's original treatment refers to distance-preserving motions of a sphere about its center, we briefly formalize the relationship between these two settings, so that we can focus once and for all on one setting, and so that results about either one can be made to correspond systematically with one another.

We denote by $\mathbf{O}(n)$ the orthogonal linear transformations of $\mathbb{R}^n$, i.e., the group of linear transformations of $\mathbb{R}^n$ that preserve the standard inner product of every pair of points. We will denote by $\mathbf{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$, and for $\mathbf{x}, \mathbf{y} \in \mathbf{S}^{n-1}$ we will denote by $\angle(\mathbf{x}, \mathbf{y})$ the angle between them, measured in radians. We note that by definition of radian measure, $\angle(\mathbf{x}, \mathbf{y})$ is the usual spherical distance between $\mathbf{x}$ and $\mathbf{y}$; that is, it is the length of the shorter of the two geodesic (i.e., great circle) segments joining them (or π if they are antipodal). In particular, this makes $\mathbf{S}^{n-1}$ into a metric space. If $\mathbf{x}$ is any nonzero vector in $\mathbb{R}^n$, we will write $\hat{\mathbf{x}} := \frac{\mathbf{x}}{||\mathbf{x}||}$ for its normalization, a point of $\mathbf{S}^{n-1}$, and we note that if $\mathbf{x}$ and $\mathbf{y}$ are both nonzero then $\angle(\mathbf{x}, \mathbf{y})$, the angle between them, is by definition $\angle(\hat{\mathbf{x}}, \hat{\mathbf{y}})$. Finally, we will denote by $\mathbf{Rot}^n$ the group of isometries of $\mathbf{S}^{n-1}$.

If $\mathbf{T}$ is any map of $\mathbf{S}^{n-1}$ to itself, there is a canonical way to extend $\mathbf{T}$ to a self-map of $\mathbb{R}^n$ that we shall denote by $\hat{\mathbf{T}}$; namely, we define $\hat{\mathbf{T}}(\mathbf{0}) = \mathbf{0}$, and if $\mathbf{x}$ is a nonzero vector, then $\hat{\mathbf{T}}(\mathbf{x}) := ||\mathbf{x}|| \, \mathbf{T}(\hat{\mathbf{x}})$, where $\hat{\mathbf{x}} = \mathbf{x}/||\mathbf{x}|| \in \mathbf{S}^{n-1}$ is the normalization of x. We call $\hat{\mathbf{T}}$ the *conical extension of* $\mathbf{T}$, and we note the obvious fact that it is norm preserving. Note that if $\mathbf{T}$ is the restriction of an orthogonal map $\mathbf{R}$ of $\mathbb{R}^n$ to the sphere, then $\hat{\mathbf{T}}(\mathbf{x}) = ||\mathbf{x}|| \, \mathbf{T}(\hat{\mathbf{x}}) = ||\mathbf{x}|| \, \mathbf{R}(\hat{\mathbf{x}}) = \mathbf{R}(||\mathbf{x}|| \, \hat{\mathbf{x}}) = \mathbf{R}(\mathbf{x})$, that is, conical extension recovers an orthogonal map from its restriction to the sphere. Next note that if $\mathbf{T} \in \mathbf{Rot}^n$ then for any nonzero $\mathbf{x}$ and $\mathbf{y}$, $\angle(\hat{\mathbf{T}}\mathbf{x}, \hat{\mathbf{T}}\mathbf{y}) = \angle(\mathbf{T}\hat{\mathbf{x}}, \mathbf{T}\hat{\mathbf{y}}) = \angle(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \angle(\mathbf{x}, \mathbf{y})$. It is now easy to see that restriction to the sphere is actually an isomorphism of $\mathbf{O}(n)$ with $\mathbf{Rot}^n$.

Lemma. *A transformation* $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ *that preserves inner products is necessarily linear and hence an orthogonal transformation.*

Proof. Let $\mathbf{e}_i$, $i = 1, 2, \ldots, n$ denote the standard basis for $\mathbb{R}^n$, so for $\mathbf{x} \in \mathbb{R}^n$, $x_i := \langle x, \mathbf{e}_i \rangle$ are the components of $\mathbf{x}$. We will show that $\mathbf{Tx}$ depends linearly on the x_i. If $\mathbf{e}'_i := \mathbf{T}(\mathbf{e}_i)$, then by assumption $\langle \mathbf{e}'_i, \mathbf{e}'_j \rangle = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$, so the $\mathbf{e}'_i$ are an orthonormal basis and hence $\mathbf{Tx} = \sum_{i=1}^n \langle \mathbf{Tx}, \mathbf{e}'_i \rangle \mathbf{e}'_i = \sum_{i=1}^n \langle \mathbf{x}, \mathbf{e}_i \rangle \mathbf{e}'_i = \sum_{i=1}^n x_i \mathbf{e}'_i$. ∎

Proposition. *If* $\mathbf{T} \in \mathbf{Rot}^n$ *then* $\hat{\mathbf{T}} \in \mathbf{O}(n)$. *Thus* $\mathbf{R} \mapsto \mathbf{R}|\mathbf{S}^{n-1}$ *and* $\mathbf{T} \mapsto \hat{\mathbf{T}}$ *are mutually inverse group isomorphisms between* $\mathbf{O}(n)$ *and* $\mathbf{Rot}^n$.

Proof. We have seen that $\hat{\mathbf{T}}$ always preserves norms and, since $\mathbf{T} \in \mathbf{Rot}^n$, $\hat{\mathbf{T}}$ also preserves angles. Since $\langle \mathbf{x}, \mathbf{y} \rangle = ||\mathbf{x}|| \, ||\mathbf{y}|| \cos \angle(\mathbf{x}, \mathbf{y})$, it follows that $\hat{\mathbf{T}}$ preserves inner products, and so by the above Lemma it is an orthogonal transformation. ∎

Thus the orthogonal transformations of $\mathbb{R}^3$ (or the orthogonal matrices that represent them with respect to some basis) and the isometries of the sphere are just different ways of seeing the same thing.

We denote by $\mathbf{Rot}^n_+$ the connected component of the identity in $\mathbf{Rot}^n$. This corresponds to the subgroup $\mathbf{SO}(n)$ of $\mathbf{O}(n)$ of elements having determinant $+1$. We will call elements of this component *rotations* or *rigid motions* of $\mathbf{S}^{n-1}$ (about its center). Both terms are historical and still used widely in both physics and mathematics. It was clearly this concept that Euler had in mind when in his original statement of

his theorem he said: "Quomodoqunque sphaera circa centrum suum convertatur" ("In whatever way a sphere is rotated around its own center").

A third equivalent notion deserves passing mention as well, for both historical and mathematical context. If we consider the group of Euclidean motions of $\mathbb{R}^3$, i.e., isometries with respect to the standard metric, $\rho(x, y) := ||x - y||$, the isotropy subgroup fixing the origin is also isomorphic to $\mathbf{O}(3)$ and $\mathbf{Rot}^3$. We note that the subspaces identified in the linear algebraic setting, e.g., planes and lines through the origin in $\mathbb{R}^3$, correspond to great circles and antipodal points on $\mathbf{S}^2$, respectively, and rays in $\mathbb{R}^3$ correspond to points on the sphere. Henceforth we will implicitly identify these different frameworks while, without loss of generality, we explicitly remain in Euler's context of transformations of the sphere.

Next we list the four ways a point may behave under a distance-preserving mapping $\mathbf{T}$ of a sphere to itself. A point p is *fixed* by $\mathbf{T}$ if $\mathbf{T}(p) = p$, which implies that $\mathbf{T}^{-1}(p) = p$. A point p is *reversed* by $\mathbf{T}$ if $\mathbf{T}(p) = -p$, the antipodal point of p, which implies that $\mathbf{T}^{-1}(p) = -p$. (Here, the notation $-p$ indicates the unique farthest point on the sphere from p, and does not imply a linear structure.) We collectively call the fixed and reversed points of $\mathbf{T}$ its characteristic directions, consistent with terminology from linear algebra. A point p is *swapped* by $\mathbf{T}$ if p is not fixed under $\mathbf{T}$ and $\mathbf{T}(p) \neq -p$, but $\mathbf{T}(p) = \mathbf{T}^{-1}(p)$. A point p is *generic* (with respect to $\mathbf{T}$) if $\mathbf{T}(p) \neq \mathbf{T}^{-1}(p)$—which implies that p is not fixed. In the language of dynamical systems, a generic point has period > 2, swapped and reversed points have period 2, and fixed points have period 1, and this is clearly exhaustive.

Even the behavior of two pairs of antipodal points does not uniquely determine a rigid motion of the sphere. For example, if we find two such pairs a quarter-circumference apart, one pair fixed and the other reversed, they could be the axis of a nongeneric rotation and an antipodal pair on its equator, or alternatively, they could be an antipodal pair on the equator of a reflection and its axis. The situation is settled by the uniqueness theorem below. A better known prototype comes from plane geometry, where it is an elementary but important fact that a Euclidean motion is determined by how it maps any three noncollinear points. The following theorem states that the same principle holds for isometries of the sphere if we call three points on a sphere *noncollinear* if they are not all contained in any one great circle. In particular, no two of three noncollinear points can be antipodal points. In the framework of linear transformations, this condition guarantees linear independence of the three associated vectors, and in that setting it is well known that their images determine a linear transformation uniquely; but here we derive the result from the kind of elementary geometric considerations Euler used.

Uniqueness Theorem. *An orthogonal transformation of $\mathbb{R}^3$ is uniquely determined by how it maps any three noncollinear points on a sphere. That is, if $\mathbf{T}_1$ and $\mathbf{T}_2$ are orthogonal transformations and P_1, P_2, P_3 any three noncollinear points on a sphere centered at the origin then $\mathbf{T}_1(P_j) = \mathbf{T}_2(P_j)$, $j = 1, 2, 3$ implies $\mathbf{T}_1 = \mathbf{T}_2$.*

Proof. We will show that if $\mathbf{T}$ is orthogonal and $\mathbf{T}(P_j) = P_j$ for $j = 1, 2, 3$, then $\mathbf{T}$ must be the identity. The theorem follows immediately by considering $\mathbf{T} = \mathbf{T}_2^{-1}\mathbf{T}_1$. Let P_1 and P_2 be nonantipodal points fixed by $\mathbf{T}$ and q any point on the sphere. If the distance from P_j to q is zero or a half-circumference for $j = 1$ or 2, then preservation of distance immediately shows that q is fixed by $\mathbf{T}$. Otherwise, taking P_i as center and the distance of P_i to q as radius gives two circles C_i with distinct nonantipodal centers. Unless q is on the great circle l containing P_1 and P_2 (in which case it is again immediate that q is fixed) these circles intersect in two distinct points q_1 and q_2, and

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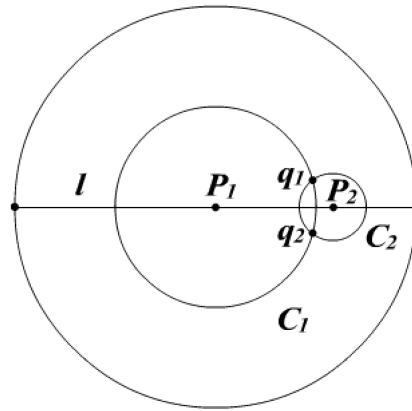
Figure 2.

either $q = q_1$ or $q = q_2$. Since P_1 and P_2 are fixed and distances are preserved, q_1 and q_2 must either be fixed or swapped by **T**. Since any point that is equidistant from q_1 and q_2 lies on l, and P_3 is fixed and does not lie on l, preservation of the unequal distances from q_1 and q_2 to P_3 forces both q_1 and q_2 to be fixed. ∎

   Our next step is to convert Euler's constructions carefully into proofs of existence of a fixed or reversed axis for the case of generic rotations (proper or improper) when a generic point can be found, and for the case of nongeneric reflections (in an equator or axis) when a swapped point can be found. After that, it becomes a simple task to combine them with our uniqueness theorem for a complete Eulerian analysis of orthogonal transformations of $\mathbb{R}^3$.

   We begin our analysis of distance-preserving transformations **T** of $\mathbb{R}^3$ about a fixed center C by assuming that there is a point whose image and preimage are distinct. We recall that such a point is called generic with respect to **T** and that **T** is called generic if there exists such a point. If A is a generic point for **T**, let $a := \mathbf{T}(A)$, and $\alpha := \mathbf{T}^{-1}(A)$. We can modify the constructions of Euler and Whittaker using these three distinct points A, a, and α to define a diameter whose points are either fixed if **T** is proper or reversed if **T** is improper. We will then develop alternative constructions to handle exceptional cases.

Geometric Proof of Euler's Theorem: Generic Case. *Let* **T** *be a distance-preserving transformation of a sphere to itself, and let A be a generic point on S. Then there exist antipodal points p_1 and p_2 such that either $\mathbf{T}(p_1) = p_1$ and $\mathbf{T}(p_2) = p_2$, or $\mathbf{T}(p_1) = p_2$ and $\mathbf{T}(p_2) = p_1$. Furthermore, any point on S that is not p_1 or p_2 is also generic.*

Remark. Although we cannot rely on it for the proof, it may help to keep in mind the following picture it implies: either α, A, and a are on a common latitude circle about the axis of a rotation by θ, where $0 < \theta < \pi$, or α and a are on a common latitude circle and A is on the opposite latitude circle, and in either case, the longitude of A is halfway between the longitudes of α and a.

Proof. (See Figures 3 and 4.) Let l be the perpendicular bisector of α and a, and $\mathbf{T}(l)$ its image under **T**. Since **T** preserves distances, $\mathbf{T}(l)$ is the perpendicular bisector of $\mathbf{T}(\alpha) = A$ and $\mathbf{T}(a)$, and therefore it is also a great circle. Since $d(\alpha, A) = d(\mathbf{T}(\alpha),$
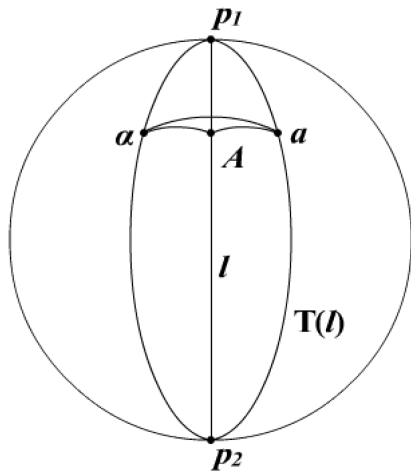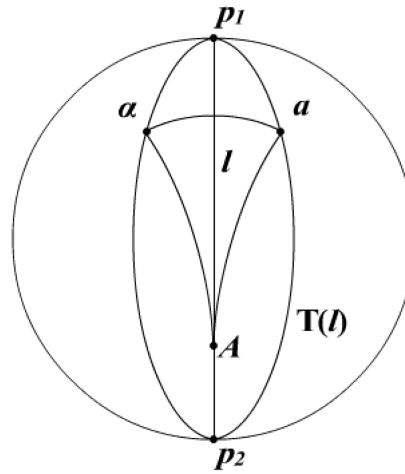
Figure 3.

Figure 4.

$\mathbf{T}(A)) = d(A, a)$, we have $A \in l$. This means $a = \mathbf{T}(A) \in \mathbf{T}(l)$, and since $a \notin l$, $\mathbf{T}(l) \neq l$ and there are two distinct antipodal intersection points of l with $\mathbf{T}(l)$, p_1, and p_2. $\mathbf{T}$ maps the antipodal points $p_1, p_2 \in l$ to antipodal points $\mathbf{T}(p_1), \mathbf{T}(p_2) \in T(l)$. Because p_1 and p_2 are the intersection points of l with $\mathbf{T}(l)$, they also are antipodal points on $\mathbf{T}(l)$.

We will now show that either $\mathbf{T}(p_1) = p_1$ and $\mathbf{T}(p_2) = p_2$ and T is proper (Figure 3), or else $\mathbf{T}(p_1) = p_2$ and $\mathbf{T}(p_2) = p_1$ and T is improper (Figure 4). To do so, we will show that for $j = 1$ and $j = 2$, triangles $p_j a \mathbf{T}(p_j)$ and $\mathbf{T}(p_j) A p_j$ are congruent, as represented symbolically in Figure 5(a).
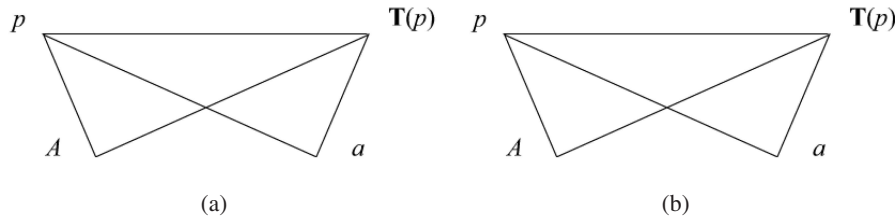

(a)

(b)

Figure 5.

Since p_1 and p_2 are both on the bisector of α and a, $d(p_j, \alpha) = d(p_j, a)$, and applying $\mathbf{T}$ to the left pair, $d(\mathbf{T}(p_j), A) = d(p_j, a)$. Since $\mathbf{T}$ preserves distance and $a = \mathbf{T}(A)$, we find $d(\mathbf{T}(p_j), a) = d(p_j, A)$. By simple symmetry, $d(\mathbf{T}(p_j), p_j) = d(p_j, \mathbf{T}(p_j))$, which is the third side congruence needed to shows that triangles $p_j a \mathbf{T}(p_j)$ and $\mathbf{T}(p_j) A p_j$ are congruent. But since p_j, a, and $\mathbf{T}(p_j)$ all lie on the great circle $\mathbf{T}(l)$, triangle $p_j a \mathbf{T}(p_j)$ is degenerate, and it is not hard to check this can only be the case if triangle $\mathbf{T}(p_j) A p_j$ also is degenerate. In other words, $\mathbf{T}(p_j)$ lies on l as well as $\mathbf{T}(l)$, i.e., $\mathbf{T}(p_1) = p_1$ and by antipodality $\mathbf{T}(p_2) = p_2$, or $\mathbf{T}(p_1) = p_2$ and $\mathbf{T}(p_2) = p_1$.

We may also use $d(a, A) = d(A, a)$ to invoke a different set of congruent triangles, $p_j A a$ and $\mathbf{T}(p_j) a A$ (Figure 5(b)), and show that our construction agrees with Euler's in the proper case. When $\mathbf{T}(p_1) = p_1$ this congruence shows that (Figure 3) triangle $p_1 A a$ is isosceles and the great circle $T(l)$ containing a and p_1 does make "the same angle at a with the arc Aa as l makes with the arc Aa at A" as in the original construc-

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tion. Euler's and Whittaker's constructions must be modified to work in the improper case, since the axis will *not* be on the bisector of A and a.

To show that any point on S other than p_1, p_2 is generic (and hence that if a generic point exists we are guaranteed to find one among any two nonantipodal points) we use our uniqueness theorem for orthogonal transformations of $\mathbb{R}^3$ proved earlier to confirm that the transformation $\mathbf{T}$ is *the* rotation about the axis $p_1 p_2$ by an angle other than 0 or π, perhaps followed by a reflection in its equator.

When $\mathbf{T}(p_1) = p_1$ and $\mathbf{T}(p_2) = p_2$, we can construct a proper rotation $\mathbf{R}$ about $p_1 p_2$ by an angle $\theta \neq 0, \pi$ so that $\mathbf{R}(p_1) = p_1$, $\mathbf{R}(\alpha) = A$, and $\mathbf{R}(A) = a \neq \alpha$, and since p_1, α, A satisfy the conditions of the uniqueness theorem, $\mathbf{T} = \mathbf{R}$ and all points on the sphere other than p_1 and p_2 are generic. When $\mathbf{T}(p_1) = p_2$ and $\mathbf{T}(p_2) = p_1$ we make the analogous argument with the improper orthogonal transformation $\mathbf{R}'$ that rotates about $p_1 p_2$ by an angle $\theta \neq 0, \pi$, and then reflects in its equator, so that $\mathbf{R}'(p_1) = p_2$, $\mathbf{R}'(\alpha) = A$, and $\mathbf{R}'(A) = a \neq \alpha$. $\mathbf{T} = \mathbf{R}'$ and again every point on the sphere other than p_1 and p_2 is generic. ∎
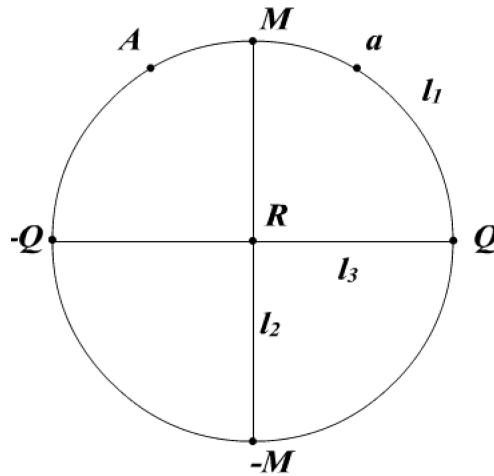

Figure 6.

The next case we treat is when a swapped point exists.

Geometric Proof of Euler's Theorem: Nongeneric Case. *Let* $\mathbf{T}$ *be a distance-preserving transformation of a sphere to itself, and let* A *be a swapped point of* $\mathbf{T}$. *Then there exist antipodal points* p_1 *and* p_2 *such that either* $\mathbf{T}(p_1) = p_1$ *and* $\mathbf{T}(p_2) = p_2$ *and the equator* ϵ *bisecting* p_1 *and* p_2 *consists of reversed points, or else* $\mathbf{T}(p_1) = p_2$ *and* $\mathbf{T}(p_2) = p_1$ *and the equator* ϵ *bisecting* p_1 *and* p_2 *consists of fixed points. Further, any point on* S *other than* p_1 *and* p_2 *or the bisecting equator is also a swapped point.*

Remark. If the points of ϵ are reversed points, then $\mathbf{T}$ is a (proper) rotation by π radians about $p_1 p_2$. If the points or ϵ are fixed, $\mathbf{T}$ is the reflection in ϵ (improper). Although we cannot rely on it for the proof, it may help to keep in mind the picture that it implies (Figure 6): either A and a have the same latitude and opposite longitudes with respect to the axis of a rotation by π radians (the axis from M to $-M$ in the figure), or they have the same longitudes and opposite latitudes with respect to the equator of some reflection (the equator bisecting Q and $-Q$ in the figure).

Proof. Since A and a are nonantipodal, there is a unique great circle, l_1, containing A and a, and because **T** takes great circles to great circles, it takes l_1 to the great circle containing $\mathbf{T}(A) = a$ and $\mathbf{T}(a) = A$, i.e., to itself. Now a distance-preserving transformation maps the perpendicular bisector of any two distinct points to the perpendicular bisector of their images. Since A and a are swapped, **T** also maps l_2, the perpendicular bisector of A and a, to itself. Thus **T** takes the intersection of the two perpendicular great circles l_1 and l_2 to itself. This consists of two points, the midpoint, M, of A and a, and its antipodal point $-M$, distinguished by the fact that M is closer than $-M$ to both A and a. These points must be individually fixed since $d(\mathbf{T}(M), a) = d(\mathbf{T}(M), \mathbf{T}(A)) = d(M, A) < d(-M, a)$. Then since M and $-M$ are fixed, **T** also maps l_3, the equator bisecting M and $-M$, to itself. This great circle is perpendicular to both l_1 and l_2 since it bisects pairs of points on both. We denote the antipodal points of intersection of l_3 with l_1 by Q and $-Q$, and of l_3 with l_2 by R and $-R$. Since $\mathbf{T}(l_j) = l_j$, $j = 1, 2, 3$, we know that the set $\{Q, -Q\}$ is mapped to itself. If both points were fixed, the circle l_1 containing M, Q, and $-Q$ could only contain fixed points, but since it also contains A and a which are swapped, we deduce that Q and $-Q$ must be reversed. For the same reason, we know that the set $\{R, -R\}$ is mapped to itself.

If R is a reversed point, the rotation by π about the axis containing M and $-M$ fixes M and swaps Q and R with their antipodal points, so by the uniqueness theorem, **T** is equivalent to this rotation, and all points other than the fixed points M and $-M$, and points on l_3, which consists entirely of reversed points, will be swapped points.

If R is a fixed point, the reflection in l_2, the equator bisecting Q and $-Q$ and containing M, $-M$, R, and $-R$, fixes M and R and swaps Q with its antipodal point, so by the uniqueness theorem, **T** is equivalent to this reflection, and all points other than the fixed great circle l_2 and the reversed points Q and $-Q$ will be swapped points. ∎

We are now ready to complete the proof of the following theorem.

Theorem. *Let* **T** *be a distance-preserving transformation of a sphere to itself. Then* **T** *may be classified and its characteristic directions identified according to one of the following six possibilities, using no more than three noncollinear points, and their images and preimages under* **T**.

1. **T** *is the identity,*
2. **T** *is the antipodal map,*
3. **T** *is a rotation by π about an axis,*
4. **T** *is a reflection across an equator,*
5. **T** *is a generic rotation about an axis,*
6. **T** *is a generic rotation about an axis followed by a reflection across its equator.*

Cases 1, 3, and 5 are proper, and cases 2, 4, and 6 are improper. If **T** *is proper, there is at least one pair of antipodal fixed points, and if* **T** *is improper, there is at least one pair of antipodal reversed points.*

Proof. If no generic or swapped points exist, then every point on the sphere must be either fixed or reversed. Since distances are preserved, a reversed point can only be found a quarter-circumference from a fixed point, and from this if every point is either fixed or reversed, in fact all points must be fixed, and **T** is the identity (proper), or all points must be reversed, and **T** is the antipodal map (improper). Otherwise, generic points exist or swapped points exist, and the above theorems guarantee that these cases are

mutually exclusive. If a generic point exists, the theorem for the generic case tells us how to find the fixed or reversed axis, and classify **T** as a proper or improper generic rotation about that axis. If instead a swapped point exists, the theorem for the nongeneric case tells us how to find the fixed axis and reversed equator classifying **T** as a proper rotation by π about that axis, or the fixed equator and reversed axis classifying **T** as an improper reflection across that equator. ∎

Remark. Algorithmically, we know that most orthogonal transformations **T** are generic. In that case, unless we chose one of the two fixed or reversed points, the very first point whose image and preimage we examine will immediately classify **T** and find its axis. If the first point we examine is not generic but swapped, it still suffices for classification and finding the characteristic directions. If it is neither, we can examine a second point one-quarter circumference from the first. If it is not generic, then **T** cannot be generic, since the only nongeneric points of a generic **T** are antipodal. If it is not swapped either, we can sample one more point a quarter circumference from each of the first two. Each of these three will be either fixed or reversed, and the particular combination tells us that **T** is the identity or the antipodal map, or that **T** is a nongeneric rotation or reflection, for which two of the points are on its equator and the other on its axis.

5. OTHER PROOFS OF EULER'S THEOREM. Finally, we survey briefly some classic approaches to proofs of Euler's Theorem based on several areas of modern mathematics: linear algebra, topology, and differential geometry and Lie theory. To anyone familiar with one of these fields, the corresponding proof will seem succinct, natural, and intuitive, and so "superior" to Euler's proof based on spherical geometry, given above. But this is analogous to comparing the often complicated geometric proofs that Isaac Newton gives in his *Principia* with the slick differential-equations-based proofs of the same facts in a modern treatise on classical mechanics. In both cases, one should keep some historical perspective and remember that the succinctness and elegance of the modern approaches is only possible because they are based on a quite massive mathematical infrastructure that was built up over many years by generations of mathematicians.

1. Approach Based on Linear Algebra. Perhaps the best-known proof of Euler's Theorem is based on the theory of eigenvalues and eigenvectors of a matrix and so, via the determinant, on the characteristic polynomial of the matrix. Once one has understood these admittedly somewhat advanced linear algebra concepts, the proof is short and transparent. Here in brief is one version.

- If **A** is a matrix, then nontrivial solutions of $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ correspond to eigenvalues of **A**, namely roots of the characteristic polynomial of **A**: $\chi_{\mathbf{A}}(\lambda) := \det(\mathbf{A} - \lambda\mathbf{I})$. Note that the the constant term $\chi(0) = \det(A)$ is the product of the eigenvalues.
- The characteristic polynomial of a real 3×3 matrix A is a cubic with real coefficients, so there are two possible case; either:

    (R) $\chi_{\mathbf{A}}$ has three real roots, r_1, r_2, and r_3, or

    (C) $\chi_{\mathbf{A}}$ has a real root, r_1, and distinct complex conjugate roots, r_2 and $r_3 = r_2^*$.

- Any eigenvalue of any orthogonal matrix has complex modulus 1. In case (R), this says $r_j = \pm 1$, $j = 1, 2, 3$, and in case (C) it says $r_2 r_3 = 1$.
- Since the determinant of a proper orthogonal matrix **A** is $+1$, the product of its eigenvalues is $+1$, and in case (R) this says they are either $\{+1, +1, +1\}$ or $\{+1, -1, -1\}$.

In case (C) it says that $r_1 = 1$. Since in both cases $+1$ is an eigenvalue, there is always a nontrivial solution to $\mathbf{Av} = \mathbf{v}$.

- Similarly, if $\mathbf{A}$ is an improper orthogonal transformation, then the product of its eigenvalues is -1, and in case (R) this says that they are either $\{+1, +1, -1\}$ or $\{-1, -1, -1\}$. In case (C) the real root is $r_1 = -1$. Thus -1 is always an eigenvalue, so there is a nontrivial solution to $\mathbf{Av} = -\mathbf{v}$. ∎

Remark. This proof can be generalized to a rotation $\mathbf{A}$ of $\mathbb{R}^n$ by observing that its real eigenvalues are ± 1 and the complex ones come in complex conjugate pairs. Then $\mathbb{R}^n$ can be written as a direct sum of orthogonal eigenspaces, and the eigenspaces are even dimensional for each complex-conjugate pair. The -1 eigenspace must also be even dimensional (and may also be viewed as the sum of two-dimensional eigenspaces on which $\mathbf{A}$ acts as a half-turn rotation). Therefore, if n is odd, the 1-eigenspace must be nontrivial. In [**12**], this proof is used to find the axis in the nongeneric case. In the generic case, the axis is derived from the Euler-Rodrigues formula [**14, 15**], giving a rotation matrix in terms of an axis and angle that are assumed to exist, so it cannot be used as an existence proof.

A different formula of Rodrigues [**16**], which is equivalent to Hamilton's quaternion multiplication and appeared earlier, *can* be used to prove Euler's theorem, as noted in [**2, 3**]. Let $\mathbf{R}_j$ denote the rotation about a unit vector $\mathbf{u}_j$ by angle θ_j, and let $(c_j, s_j) = (\cos \frac{\theta_j}{2}, \sin \frac{\theta_j}{2})$. The formula states that $\mathbf{R}_2 \mathbf{R}_1 = \mathbf{R}_3$ where $c_3 = c_1 c_2 - s_1 s_2 (\mathbf{u}_1 \cdot \mathbf{u}_2)$ and $s_3 \mathbf{u}_3 = c_1 s_2 \mathbf{u}_2 + c_2 s_1 \mathbf{u}_1 + s_1 s_2 (\mathbf{u}_2 \times \mathbf{u}_1)$. If $\mathbf{R}$ is a 3×3 rotation matrix, and $\mathbf{v}$ satisfies $\mathbf{w} = \mathbf{v} \times \mathbf{Rv} \neq \mathbf{0}$, then $\mathbf{R}$ is a rotation about $\mathbf{w}$ taking $\mathbf{v}$ to $\mathbf{Rv}$, followed by a rotation about $\mathbf{Rv}$. We can also write $\mathbf{R}$ as the product of three 'Euler angles,' i.e., rotations about coordinate axes. With proper consideration of nongeneric cases, these decompositions, combined with two or three applications of Rodrigues' formula amount to a constructive proof of Euler's theorem.

2. Approach Based on Topology. A simple corollary of the Lefschetz Fixed Point Theorem is that if a compact manifold X has nonzero Euler characteristic, $\chi(X)$, then any continuous self-mapping of X that is homotopic to the identity must have a fixed point. Recall that $b_i(X)$, the ith Betti number of X, is the rank of its ith homology group and the Euler characteristic is the alternating sum $\sum_i (-1)^i b_i(X)$. Now for the sphere $\mathbf{S}^n$, $b_0 = b_n = 1$, and all other Betti numbers are zero, so $\chi(\mathbf{S}^n)$ is two or zero depending on whether n is even or odd. So we see that in particular any continuous self-map of $\mathbf{S}^{2k}$ that is homotopic to the identity map will have a fixed point. Since Euler's concept of a rigid motion of a sphere certainly included that the motion was the endpoint of a continuous family that started at the identity, Euler would have seen this as a generalization of his result. The Lefschetz theorem uses quite sophisticated topological arguments, but for the sphere there is an elegant and simpler approach that gets nearly as strong a result.

Theorem (Borsuk-Hirsch). *Let $f : \mathbf{S}^n \to \mathbf{S}^n$ be a continuous self-map of the n-dimensional sphere. If f does not have any fixed points, then it is homotopic to the antipodal map $-I$. Similarly, if there is no point p of $\mathbf{S}^n$ such that $f(p) = -p$, then f is homotopic to the identity map I.*

Proof. For $0 \leq t \leq 1$, define $\phi_t : \mathbf{S}^n \to \mathbb{R}^{n+1}$ by $\phi_t(x) = (1 - t) f(x) - tx$. If for some t and x, $\phi_t(x) = \mathbf{O}$, then $tx = (1 - t) f(x)$, so taking the norm of both sides and using $||x|| = ||f(x)|| = 1$, $t = 1 - t$ so $t = \frac{1}{2}$. Then $\frac{1}{2} x = \frac{1}{2} f(x)$ so $f(x) = x$.

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Hence, if f has no fixed points then $\Phi_t(x) := \phi_t(x)/||\phi_t(x)||$ defines a homotopy between f and $-I$. Changing the definition of ϕ_t to $\phi_t(x) := (1-t)f(x) + tx$, the same argument shows that if there is no $x \in \mathbf{S}^n$ with $f(x) = -x$, then $\Phi_t(x)$ defines a homotopy between f and I. ∎

Suppose $n = 2k$ is even and write $\mathbb{R}^{2k}$ as the orthogonal direct sum of k two-dimensional subspaces. If we simultaneously rotate each of them through an angle t, then as t varies from 0 to π we get a homotopy of I with $-I$. Thus the identity map and the antipodal map of S^{2k-1} are homotopic. However, the degree of the identity map of S^{2k} is $+1$ while the degree of the antipodal map of S^{2k} is -1. Since homotopic maps of an n-dimensional manifold have the same degree, it follows that for an even-dimensional sphere the identity map and the antipodal map are not homotopic, and so by the above theorem *any continuous map $f : S^{2k} \to S^{2k}$ either has a fixed point p or a point p that is mapped to its antipodal point $-p$.* Of course, if f is an orthogonal transformation, it has degree $+1$ if it is proper, and hence has a fixed point, and it has degree -1 if it is improper, and hence there is a point p that is mapped to its antipodal point. See [**10**] for more details on this approach.

3. Approach Based on Differential Geometry and Lie Theory. Euler's Theorem is an easy consequence of the following three well-known results from Riemannian geometry and Lie group theory. (See the discussion below.)

Hopf-Rinow-DeRham Theorem. *A Riemannian manifold M is geodesically complete if and only if it is metrically complete. Hence, if M is compact, then every pair of points of M can be joined by a geodesic parameterized by arclength.*

Proposition. *A compact Lie group always admits a Riemannian metric that is invariant under both left and right translation. Moreover, for any such metric, the geodesics parametrized proportionally to arclength and starting at the identity are exactly the one-parameter subgroups. Hence, every element of a compact Lie group lies on some one-parameter subgroup.*

Lemma. *The one-parameter subgroups of $\mathbf{SO}(n)$ are of the form $\mathbf{g}(t) = \exp(t\mathbf{A})$, where $\mathbf{A}$ is some $n \times n$ skew-symmetric matrix. That is, if $\mathbf{v} \in \mathbb{R}^n$, then $\mathbf{x}(t) = \mathbf{g}(t)\mathbf{v}$ is the unique solution of the linear differential equation $\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) = \mathbf{v}$. Hence, if $\mathbf{A}\,\mathbf{v} = \mathbf{O}$, then $\mathbf{g}(t)\mathbf{v} = \mathbf{v}$ for all t.*

For proofs see [**17**], pages 342, 401, and 378.

Now suppose $g \in \mathbf{SO}(3)$. By the above proposition and lemma there is a 3×3 skew-symmetric matrix $\mathbf{A}$ and a real t such that $\mathbf{g} = \exp(t\mathbf{A})$, and as we saw in Section 2, if

$$\mathbf{A} = \begin{pmatrix} 0 & a_{12} & -a_{31} \\ -a_{12} & 0 & a_{23} \\ a_{31} & -a_{23} & 0 \end{pmatrix},$$

and we let $\mathbf{v} := (a_{23}\ a_{31}\ a_{12})^T$, then $\mathbf{A}\mathbf{v} = \mathbf{O}$, so $\mathbf{g}\mathbf{v} = \mathbf{v}$, i.e., $\mathbf{v}$ is an axis for $\mathbf{g}$. (If $\mathbf{v} = \mathbf{O}$, then $\mathbf{A}$ is the zero matrix and so $\mathbf{g}$ is the identity.) This proves Euler's Theorem. ∎

DEDICATION AND ACKNOWLEDGMENTS. Dedicated to the memory of Leonhard Euler, "the master of us all," on the occasion of the 300th anniversary of his birth.

The authors gratefully acknowledge important suggestions from a reviewer that resulted in significant improvements in the final manuscript.

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