EQUIVALENCE OF NEARBY DIFFERENTIABLE ACTIONS OF A COMPACT GROUP

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In this note we will be concerned with the proof and consequences of the following fact: if ϕ_0 is a differentiable action of a compact Lie group on a compact differentiable manifold M, then any differentiable action of G on M sufficiently close to ϕ_0 in the C^1-topology is equivalent to ϕ_0.

1. **Notation.** In what follows differentiable means class C^∞. If M and V are differentiable manifolds, $\mathfrak{M}(M, V)$ is the space of differentiable maps of M into V in the C^K-topology where K is a positive integer or ∞ fixed throughout. We denote by Diff (M) the group of automorphisms of M topologized as a subspace of $\mathfrak{M}(M, M)$. As such it is a topological group. $\mathfrak{D}(M)$ is the subgroup of Diff (M) consisting of diffeomorphisms which are the identity outside of some compact set and $\mathfrak{D}_0(M)$ is the arc component of i_M, the identity map of M, in $\mathfrak{D}(M)$. If M is compact $\mathfrak{D}(M)$ is locally arcwise connected and $\mathfrak{D}_0(M)$ is open in $\mathfrak{D}(M)$ and in fact in $\mathfrak{M}(M, M)$. For a definition of the C^K-topology and a proof of the statements made above, see [6]. If G is a Lie group we denote by $\mathfrak{A}(G, M)$ the space of differentiable actions of G on M, i.e. continuous homomorphisms of G into Diff (M), topologized with the compact-open topology. If $\phi: g \to g^\phi$ is an element of $\mathfrak{A}(G, M)$ then by a theorem of D. Montgomery [2] $\tilde{\phi}: (g, m) \to g^\phi m$ is an element of $\mathfrak{M}(G \times M, M)$. Given $\phi \in \mathfrak{A}(G, M)$ and $f \in$ Diff (M) then ϕ composed with the inner automorphism of Diff (M) defined by f is another element $f\phi$ of $\mathfrak{A}(G, M)(g^{f\phi} = fg^\phi f^{-1})$. Clearly $(f, \phi) \to f\phi$ is jointly continuous[2] and defines an action of Diff (M) on $\mathfrak{A}(G, M)$. We henceforth consider $\mathfrak{A}(G, M)$ as a Diff (M)-space and, *a fortiori* as a $\mathfrak{D}(M)$ and $\mathfrak{D}_0(M)$-space. Note that the orbit space $\mathfrak{A}(G, M)/$Diff (M) is just the set of equivalence classes of actions of G on M.

2. **Statement of main theorem and consequences.** The following theorem will be proved in §3.

THEOREM A. *If M is a compact differentiable manifold and G is a compact Lie group then the $\mathfrak{D}_0(M)$-space $\mathfrak{A}(G, M)$ admits local cross sections; i.e. given $\phi_0 \in \mathfrak{A}(G, M)$ there is a neighborhood U of ϕ_0 in*

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[2] This follows from the proposition in [6, §1].

$\mathfrak{A}(G, M)$ and a continuous map χ: $U \to \mathfrak{D}_0(M)$ such that $\chi(\phi_0) = i_M$ and $\chi(\phi)\phi_0 = \phi$.

COROLLARY 1. *If ϕ_t is a continuous arc in $\mathfrak{A}(G, M)$ then there is a continuous arc f_t in $\mathfrak{D}_0(M)$ such that $f_0 = i_M$ and $\phi_t = f_t\phi_0$.*

REMARKS. Corollary 1 was proved in [7] by the author and T. E. Stewart under the added hypothesis that $(g, m, t) \to \tilde{\phi}_t(g, m)$ was jointly differentiable in all three variables. It was shown there by counter-example that Corollary 1 is invalid if we consider continuous rather than differentiable actions or if we drop either of the conditions that G or M be compact. It follows that all these conditions are also necessary for the validity of Theorem A.

Using that $\mathfrak{D}_0(M)$ is locally arcwise connected:

COROLLARY 2. *$\mathfrak{A}(G, M)$ is locally arcwise connected. If $\phi_0 \in \mathfrak{A}(G, M)$ then its orbit under $\mathfrak{D}_0(M)$ is its arc component in $\mathfrak{A}(G, M)$ hence an open set, and its orbit under $\mathfrak{D}(M)$ (i.e. the class of actions equivalent to ϕ_0) is also open and so a union of arc components. Moreover if $\Delta = \{f \in \mathfrak{D}(M) \mid f\phi_0 = \phi_0\}$ is the group of automorphisms of the differentiable G-space (M, ϕ_0) then $f\Delta \to f\phi_0$ is a homeomorphism of $\mathfrak{D}(M)/\Delta$ onto $\mathfrak{D}(M)\phi_0$.*

Since $\mathfrak{A}(G, M)$ is separable metric and each equivalence class is open:

COROLLARY 3. *There are at most countably many inequivalent differentiable actions of G on M.*

REMARKS. It seems likely that by modifying a construction of R. Bing [1] one could construct uncountably many continuous actions of Z_2 on S^3 with fixed point sets pairwise inequivalently embedded 2-spheres. These actions would of course all be inequivalent.

The following extension theorem generalizes Theorem A. On the other hand it is an easy consequence of Theorem A above and Theorem B of [6].

THEOREM B. *Let H be a Lie group, W a differentiable manifold (neither necessarily compact), G a compact subgroup of H, and M a compact submanifold of W. Let $\psi_0 \in \mathfrak{A}(H, W)$ such that M is invariant under $\psi_0 \mid G$ and let $\phi_0 \in \mathfrak{A}(G, M)$ be the induced action of G on M. Then given any neighborhood $\mathfrak{O}$ of M in W there exists a neighborhood U of ϕ_0 in $\mathfrak{A}(G, M)$ and a map ψ: $U \to \mathfrak{A}(H, W)$ such that $\psi(\phi_0) = \psi_0$, $\psi(\phi) \mid G$ leaves M invariant and induces ϕ on M, and $\psi(\phi)$ agrees with ψ_0 outside $\mathfrak{O}$. In fact there is a continuous map χ: $U \to \mathfrak{D}_0(W)$ such that $\chi(\phi)$ is the identity outside $\mathfrak{O}$ and such that $\psi(\phi) = \chi(\phi)\psi_0$ satisfies the above conditions.*

3. **Proof of Theorem A.** By a theorem proved independently by the author [5] and G. D. Mostow [4] there exists an orthogonal representation $g \to g^{\psi}$ of G in a Euclidean vector space V and a differentiable ϕ_0-equivariant embedding $f_0\colon M \to V$. Let $\mathcal{O}$ be a tubular neighborhood of $f_0(M)$ in V with respect to the Euclidean metric. Then $\mathcal{O}$ is invariant under the representation ψ and the map $\pi\colon \mathcal{O} \to f_0(M)$ carrying a point of $\mathcal{O}$ into the unique nearest point of $f_0(M)$ is a differentiable equivariant retraction of $\mathcal{O}$ onto $f_0(M)$. Given $\phi \in \mathfrak{A}(G, M)$ define $f_\phi\colon M \to V$ by $f_\phi(m) = \int g^{-1\psi} f_0(g^\phi m)dg$ where the integral is with respect to Haar measure on G. Then (cf. [4, p. 434]) f_ϕ is ϕ-equivariant and clearly $f_{\phi_0} = f_0$. The map $F_\phi \in \mathfrak{M}(G \times M, V)$ defined by $F_\phi(g, m) = \tilde{\psi}(g^{-1}, f_0 \circ \tilde{\phi}(g, m))$ is easily seen[2] to depend continuously on $\phi \in \mathfrak{A}(G, M)$ and since $f_\phi = \int F_\phi(g, m)dg$ it follows that $\phi \to f_\phi$ is a continuous map of $\mathfrak{A}(G, M)$ into $\mathfrak{M}(M, V)$. Then for ϕ in a neighborhood U' of ϕ_0 in $\mathfrak{A}(G, M)$ $f_\phi(M) \subseteq \mathcal{O}$ so $\sigma(\phi) = f_{\phi_0}^{-1} \circ \pi \circ f_\phi \in \mathfrak{M}(M, M)$. Now $\sigma\colon U' \to \mathfrak{M}(M, M)$ is continuous[2] and clearly $\sigma(\phi_0) = i_M$. Since $\mathfrak{D}_0(M)$ is open in $\mathfrak{M}(M, M)$, for some smaller neighborhood U of ϕ_0 in $\mathfrak{A}(G, M)$ $\sigma\colon U \to \mathfrak{D}_0(M)$. Since f_ϕ, π, at f_{ϕ_0} are respectively ϕ-, π- and ϕ_0-equivariant maps into (V, ψ) it follows that $\sigma(\phi)g^\phi = g^{\phi_0}\sigma(\phi)$ or putting $\chi(\phi) = \sigma(\phi)^{-1}$, $\chi(\phi)\phi_0 = \phi$. Q.E.D.

4. **Conjugacy of neighboring compact subgroups of** Diff(M). It is suggested by Theorem A that an analogue of the Montgomery and Zippin conjugacy theorem for neighboring compact subgroups of a Lie group [3] might hold for Diff (M), i.e. that given a compact subgroup G of Diff (M) every compact subgroup of Diff (M) sufficiently close to G is conjugate in Diff (M) to a subgroup of G. This in fact is the case and was the basis of an earlier more complicated proof of Theorem A. A proof will appear elsewhere.

References

1. R. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. vol. 56 (1952) pp. 354–362.

2. D. Montgomery, *Topological groups of differentiable transformations*, Ann. of Math. vol. 46 (1945) pp. 382–387.

3. D. Montgomery and L. Zippin, *A theorem on Lie groups*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 448–452.

4. G. D. Mostow, *Equivariant embeddings in Euclidean space*, Ann. of Math. vol. 65 (1957) pp. 432–446.

5. R. S. Palais, *Imbedding of compact, differentiable, transformation groups in orthogonal representations*, J. Math. Mech. vol. 6 (1957) pp. 673–678.

6. ———, *Local triviality of the restriction map for embeddings*, Comment. Math. Helv. vol. 34 (1960) pp. 305–312.

7. R. S. Palais and T. E. Stewart, *Deformations of compact differentiable transformation groups*, Amer. J. Math. vol. 82 (1960) pp. 935–937.

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