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ON THE EXISTENCE OF SLICES FOR ACTIONS OF NON-COMPACT LIE GROUPS

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If G is a topological group then by a G-space we mean a completely regular space X together with a fixed action of G on X. If one restricts consideration to compact Lie groups then a substantial general theory of G-spaces can be developed. However if G is allowed to be anything more general than a compact Lie group, theorems about G-spaces become extremely scarce, and it is clear that if one hopes to recover any sort of theory, some restriction must be made on the way G is allowed to act. A clue as to the sort of restriction that should be made is to be found in one of the most fundamental facts in the theory of G-spaces when G is a compact Lie group; namely the result, proved in special cases by Gleason [2], Koszul [5], Montgomery and Yang [6] and finally, in full generality, by Mostow [8] that there is a "slice" through each point of a G-space (see 2.1.1 for definition). In fact it is clear from even a passing acquaintance with the methodology of proof in transformation group theory that if G is a Lie group and X a G-space with compact isotropy groups for which there exists a slice at each point, then many of the statements that apply when G is compact are valid in this case also.

In § 1 of this paper we define a G-space X (G any locally compact group) to be a Cartan G-space if for each point of X there is a neighborhood U such that the set of g in G for which $gU \cap U$ is not empty has compact closure. In case G acts freely on X (i.e., the isotropy group at each point is the identity) this turns out to be equivalent to H. Cartan's basic axiom *PF* for principal bundles in the *Seminaire H. Cartan* of 1948-49, which explains the choice of name.

In § 2 we show that if G is a Lie group then the Cartan G-spaces are precisely those G-spaces with compact isotropy groups for which there is a slice through every point.

As remarked above this allows one to extend a substantial portion of the theory of G-space that holds when G is a compact Lie group to Cartan G-spaces (or the slightly more restrictive class of proper G-spaces, also introduced in § 1) when G is an arbitrary Lie group. Part of this extension is carried out in § 4, more or less by way of showing what can be done. In particular we prove a generalization of Mostow's equivariant embed-

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ding theorem [8].

0. Notation

G will denote a locally compact group with identity e. In the second part of the paper we will restrict G to be a Lie group. If X is a space, an *action* of G on X is a homomorphism T of G into the group of homeomorphisms of X such that the function $(g, x) \to T(g)x$ from $G \times X \to X$ is continuous. A *G-space* is a completely regular space X together with a fixed action T of G on X. We will in general not explicitly mention the action T and write gx for $T(g)x$. If $g \neq e$ implies $gx \neq x$ for some $x \in X$, we say that G is *effective* on X, and if $g \neq e$ implies $gx \neq x$ for each $x \in X$, we say that G *acts freely* on X or that X is a *G-principal bundle* (in the wide sense). If $x \in X$ we write G_x for the *isotropy group* at $x = \{g \in G \mid gx = x\}$ and we write Gx for the orbit of $x = \{gx \mid g \in G\}$. More generally if $S \subseteq X$ we write GS for the *saturation* of $S = \{gs \mid g \in G, s \in S\}$. We denote the set of orbits of X under G, made into a topological space with the usual identification space topology, by X/G and we write Π_X for the orbit map of X onto X/G. In what follows we will often have occasion to consider subsets of G of the form $\{g \in G \mid gU \cap V \neq \emptyset\}$ where U and V are subsets of a G-space X. We will denote this set by $((U, V))$. A G-space X will be called *differentiable* if X is differentiable manifold and each of the maps $x \to gx$ is differentiable; *Riemannian* if in addition X is given a Riemannian structure and each of the maps $x \to gx$ is an isometry of X; and *linear* if X is a finite dimensional real vector space and each of the maps $x \to gx$ is linear.

1.1. Thin sets and Cartan G-spaces

1.1.1. DEFINITION. If U and V are subsets of a G-space X then we shall say that *U is thin relative to V* if $((U, V))$ has compact closure in G. If U is thin relative to itself then we say that *U is thin*.

Since $(gU \cap V) = g(U \cap g^{-1}V)$ it follows that if U is thin relative to V then V is thin relative to U, hence we will often simply say that U and V are relatively thin. Because $(gg_1U \cap g_2V) = g_2(g_2^{-1}gg_1U \cap V)$ it follows that if U and V are relatively thin then so are any translates g_1U and g_2V. In particular if U is thin then any pair of translates of U are relatively thin. It is clear that if U and V are relatively thin and $U' \subseteq U$, $V' \subseteq V$ then U' and V' are relatively thin. In particular a subset of a thin set is thin. Because we clearly have $((\bigcup_i U_i, V)) = \bigcup_i((U_i, V))$, any finite union of sets thin relative to V is itself thin relative to V. More generally $((\bigcup_i U_i, \bigcap_i V_i)) \subseteq \bigcup_i((U_i, V_i))$ so that if U_i is thin relative to

V_i, $i = 1, \cdots n$ then $\bigcup_{i=1}^n U_i$ is thin relative to $\bigcap_{i=1}^n V_i$. If K_1 and K_2 are compact subsets of a G-space then it is easily seen that $((K_1, K_2))$ is closed in G, hence if K_1 and K_2 are relatively thin compact sets then $((K_1, K_2))$ is compact. Finally we note that if G is compact then of course every subset of a G-space is thin. In what follows we shall use these simple properties of the concept of thinness without further mention.

1.1.2. DEFINITION. A G-space X is a *Cartan G-space* if every point of X has a thin neighborhood.

If G is compact then the notion of a Cartan G-space becomes trivial. In fact in this case every G-space satisfies the stronger condition of being proper which we will define shortly.

We now explain the choice of name. Let X be a G-principal bundle (in the wide sense) and let R be the set of pairs (x_1, x_2) in $X \times X$ for which x_1 and x_2 belong to the same orbit. Since X is principal it follows that for each $(x_1, x_2) \in R$ there is a unique element $f(x_1, x_2) \in G$ such that $x_2 = f(x_1, x_2)x_1$. Cartan (in the *Seminaire Henri Cartan* 1948-1949) has restricted the term principal bundle to principal bundles in the wide sense for which the function $f: R \to G$ is continuous. We shall call such restricted principal bundles *Cartan principal bundles*. In a general G-space the function f is not well defined and it would seem at first glance that there is no natural way of generalizing the Cartan condition. However as the reader may have guessed from our choice of terminology we have

1.1.3. THEOREM. *A G-principal bundle X is a Cartan principal bundle if and only if it is a Cartan G-space.*

PROOF. Suppose X is a Cartan principal bundle and $x \in X$. Since $f(x, x) = e$, if K is a compact neighborhood of e then there is a neighborhood U of x in X such that $f((U \times U) \cap R) \subseteq K$. Then clearly $((U, U)) \subseteq K$ so U is thin and it follows that X is a Cartan G-space.

Conversely suppose X is a Cartan G-space and let $(x_\alpha, g_\alpha x_\alpha) \to (x, gx)$. We must show that $g_\alpha \to g$. Let U be a thin neighborhood of x. We can suppose that $x_\alpha \in U$ and $g_\alpha x_\alpha \in gU$ so that $g_\alpha \in ((U, gU))$ which has compact closure. Thus if $g_\alpha \not\to g$ by passing to a subnet if necessary, we can suppose $g_\alpha \to g' \neq g$. But then since $x_\alpha \to x$, $gx = \lim_\alpha g_\alpha x_\alpha = g'x$ and hence, since G acts freely on X, $g = g'$ a contradiction. Hence g_α must converge to g. q.e.d.

We will now derive some of the elementary properties of Cartan G-spaces. It should be noted that the following propositions are standard results for G-spaces when G is compact which are not valid for arbitrary G-spaces when G is not compact.

1.1.4. PROPOSITION. *If X is a Cartan G-space then each orbit of X is closed in X* (i.e., *X/G is a T_1-space*) *and each isotropy group of X is compact.*

PROOF. Given $x \in X$ let V be a thin neighborhood of x. Clearly G_x is closed in G and since it is included in $((V, V))$ it is compact. Now let y be adherent to Gx and let U be a thin neighborbood of y. Let $g_\alpha x$ be a net in U converging to y. Fixing α_0, $(g_\alpha g_{\alpha_0}^{-1})(g_{\alpha_0}x) = g_\alpha x$ so $g_\alpha g_{\alpha_0}^{-1} \in ((U, U))$ and by passing to a subnet we can suppose that $g_\alpha g_{\alpha_0}^{-1}$ converges and hence that g_α converges say to g. Then $y = \lim g_\alpha x = gx \in Gx$ so Gx is closed. q.e.d.

Later we shall derive a considerably stronger result, namely that X/G is locally completely regular. However we now show by example that X/G need not be Hausdorff. Let X be the strip in the plane defined by $\{(x, y) \mid -1 \leqq x \leqq 1\}$. We make X into a Cartan R-principal bundle (R = additive group of real numbers) such that X/R is not Hausdorff. If $|x_0| < 1$ let $C_{(x_0, y_0)}$ be the vertical translate of the graph of the equation $y = x^2/(1 - x^2)$ which contains (x_0, y_0). We define $t(x_0, y_0)$ to be the point (x, y) on $C_{(x_0, y_0)}$ such that the length of the arc of $C_{(x_0, y_0)}$ between (x_0, y_0) and (x, y) is $|t|$ and x is greater than or less than x_0 according as t is positive or negative; i.e., (x_0, y_0) moves "counter-clockwise" along $C_{(x_0, y_0)}$ with unit velocity. To complete the definition of the action of R on X we define $t(1, y) = (1, y + t)$ and $t(-1, y) = (-1, y - t)$. It is easily seen that if K is a compact subset of X then K is thin if and only if it does not meet both of the boundary lines $x = 1$ and $x = -1$. It follows that X is a Cartan R-principal bundle. On the other hand, it is clear that the orbits $x = 1$ and $x = -1$ cannot be separated by saturated open sets, hence X/R is not Hausdorff.

LEMMA. *If X is a Cartan G-space and $x \in X$ then $g \to gx$ is an open map of G onto Gx.*

PROOF. By homogeneity it will suffice to show that if K is a neighborhood of e in G then Kx is a neighborhood of x in Gx. Suppose not. Then there is a net g_α in G such that $g_\alpha x \notin Kx$ but $g_\alpha x \to x$. Since clearly $g_\alpha x \in Kx \Longleftrightarrow g_\alpha \in KG_x$ it follows that $g_\alpha \notin KG_x$ and since KG_x is a neighborhood of G_x, that no subnet of g_α can converge to an element of G_x. Let U be a thin neighborhood of x. Since $g_\alpha x$ is eventually in U, by passing to a subnet we can suppose that $g_\alpha \in ((U, U))$ and so by again passing to a subnet we can suppose that $g_\alpha \to g$. But then $gx = \lim g_\alpha x = x$ so $g \in G_x$ a contradiction. q.e.d.

1.1.5. PROPOSITION. *If X is a Cartan G-space and $x \in X$ then the map $gG_x \to gx$ is a homeomorphism of G/G_x onto Gx.*

PROOF. Immediate from the lemma and the openness of the canonical map of G onto G/G_x.

1.1.6. PROPOSITION. *If X is a Cartan G-space and $x \in X$ then given any neighborhood U of G_x in G there is a neighborhood V of x in X such that $((V, V)) \subseteq U$.*

PROOF. We can assume that U is open and (because G_x is compact) a union of left G_x cosets. By 1.1.5 $(G - U)x$ is closed in Gx and hence (by 1.1.4) in X so we can find a neighborhood W of x such that $W \cap (G - U)x$ is empty. We can suppose that W is closed and thin. Let K be the closure of $\{g \in G - U \mid gW \cap W \neq \varnothing\}$ so that K is a compact subset of $G - U$. If $k \in K$ then $kx \in (G - U)x$ so $kx \in X - W$ and since $X - W$ is open we can find a neighborhood Q_k of k and a neighborhood V_k of x such that $Q_k V_k \subseteq X - W$. We can suppose $V_k \subseteq W$. Let $Q_{k_1}, \cdots, Q_{k_n}$ cover K and put $V = \bigcap_{i=1}^n V_{k_i}$ so that V is a neighborhood of x included in W. Now suppose $g \in ((V, V))$, i.e., $gV \cap V \neq \varnothing$. Then $gW \cap W \neq \varnothing$ hence $g \in U \cup K$. To complete the proof we will show that $g \notin K$. In fact if $g \in K$ then $g \in Q_{k_i}$ so $gV \subseteq Q_{k_i} V_{k_i} \subseteq X - W \subseteq X - V$ so that $gV \cap V = \varnothing$ a contradiction. q.e.d.

REMARK. It follows that $G_y \in U$ if $y \in V$ and hence by the conjugacy theorem of Montgomery and Zippin [7] that if G is a Lie group then all isotropy groups of points sufficiently close to x are conjugate to a subgroup of G_x. However we will give a direct proof of this fact later, using slices, and then with this we will give a simple proof of the conjugacy theorem referred to.

LEMMA. *If X is a Cartan G-space and N is the kernel of the action of G on X then X is a Cartan G/N-space.*

PROOF. Note that $N = \bigcap_{x \in X} G_x$ is compact by 1.1.4 so that a subset of X is thin when X is considered as a G/N-space if and only if it is thin when X is considered as a G-space. q.e.d.

1.1.7. PROPOSITION. *If X is a Cartan G-space then the action of G on X is a continuous and relatively open map of G into the group of homeomorphisms of X when the latter is given the topology of pointwise convergence.*

PROOF. By the lemma we can suppose that X is an effective G-space. Continuity is clear from the definition of a G-space. To prove openness let V be a neighborhood of e in G with $g_\alpha \notin V$. We will show that $g_\alpha x \to x$ for each $x \in X$ leads to a contradiction. In fact if K is a thin neighborhood of some point x_0 then since $g_\alpha x_0 \to x_0$ it follows that $g_\alpha \in ((K, K))$ for sufficiently large α and by passing to a subnet we can suppose $g_\alpha \to g$.

But then $gx = \lim g_\alpha x = x$ for all $x \in X$ so $g = e$ by effectiveness. But $g_\alpha \to e$ contradicts $g_\alpha \notin V$. q.e.d.

COROLLARY. *If X is a Cartan G-space then the set of homeomorphisms $\{x \to gx \mid g \in G\}$ is closed in the group of all homeomorphisms of X in the topology of pointwise convergence.*

PROOF. It follows from the theorem that this set of homeomorphisms is a subgroup isomorphic (as a topological group) to G/N where $N = \{g \in G \mid gx = x \text{ for all } x \in X\}$. But G/N is locally compact, and a locally compact group is closed in any containing topological group. q.e.d.

1.2. Small sets and proper G-spaces

1.2.1. DEFINITION. A subset S of a G-space X is a *small subset of X* if each point of X has a neighborhood which is thin relative to S.

Unlike thin, which is absolute, small is a relative notion, i.e., if Y is a G-space and X is an invariant subspace then a small subset of X is not necessarily a small subset of Y. Nevertheless when no confusion is likely we will often speak of a small set.

1.2.2. DEFINITION. A G-space X is *proper* if each point of X has a small neighborhood.

We now list a number of elementary facts about small sets and proper G-spaces. The proofs follow easily from the properties of thinness given at the beginning of § 1.1 and are left to the reader.

1. A subset of a small set is small.
2. A finite union of small sets is small.
3. If S is a small subset of X and K is a compact subset of X then K is thin relative to S and in fact K has a neighborhood which is thin relative to S.
4. If X is a proper G-space then every compact subset of X is small and in fact has a small neighborhood.
5. If X is a proper G-space then every compact subset of X is thin and in fact has a thin neighborhood.
6. If X is a proper G-space and K is a compact subset of X then $((K, K))$ is a compact subset of G.
7. If G is compact then every G-space is proper.

1.2.3. PROPOSITION. *A proper G-space is a Cartan G-space.*

PROOF. Let X be a proper G-space and $x \in X$. Let S be a small neighborhood of x and V a neighborhood of x thin relative to S. Then $U \cap S$ is a thin neighborhood of x. q.e.d.

The converse is false. We shall see in fact that a Cartan G-space X is proper if and only if X/G is completely regular and we know (see remark following 1.1.4) that the latter need not to the case. Nevertheless

1.2.4. PROPOSITION. *If U is a thin open set in a G-space X then GU is a proper G-space, hence every point of a Cartan G-space is contained in an invariant open set which is a proper G-space.*

PROOF. Immediate from the fact that gU and γU are relatively thin for any $g, \gamma \in G$.

1.2.5. PROPOSITION. *If X is a Cartan G-space and X/G is regular then X is a proper G-space.*

PROOF. Let $x \in X$ and choose U a thin open neighborhood of x. Then GU is a neighborhood of Gx and since X/G is regular we can find a closed invariant neighborhood W of Gx included in GU. Let $O = W \cap U$. We shall show that O is a small neighborhood of x in X. In fact if $y \in GU$, say $y \in gU$, then gU is a neighbourhood of y thin relative to U and *a fortiori* relative to O. If $y \notin GU$ then $X - W$ is a neighborhood of y and since W is invariant $((X-W, O)) = \varnothing$ so $X-W$ is thin relative to O. q.e.d.

We omit the proof of the following easy and well-known fact.

LEMMA. *If K is a compact space and M a metric space, denote by M^K the space of continuous maps of K into M metrized by* $\rho(f_1, f_2) = \operatorname{Sup}\{\rho(f_1(k), f_2(k)) \mid k \in K\}$ *so that convergence means uniform convergence on K. If X is an arbitrary space and $f: X \times K \to M$ is continuous define $f_x \in M^K$ for each $x \in X$ by $f_x(k) = f(x, k)$. Then $x \to f_x$ is a continuous map of X into M^K. Hence if M is a Banach space and μ is any Radon measure on K then $x \to \int f_x(k)d\mu(k)$ is a continuous map of X into M.*

The following replaces the well-known technique of " averaging over the group" that plays an important role in the theory of compact transformation groups.

1.2.6. PROPOSITION. *Let X be a G-space and f a continuous map of X into a linear G-space V. If the support of f is a small subset S of X then $F(x) = \int g^{-1}f(gx)d\mu(g)$ (where μ is right Haar measure on G) is an equivariant map of X into V,* i.e., *$F(gx) = gF(x)$ for all $(g, x) \in G \times X$. Moreover $F(x) = 0$ unless $x \in GS$.*

PROOF. Let $x_0 \in X$ and let U be a neighborhood of x_0 which is thin relative to S. If W is the closure of $((U, S))$ then W is compact and clearly $f(gx) = 0$ if $x \in U$ and $g \notin W$, hence $F \mid U(x) = \int_W g^{-1}f(gx)d\mu(g)$. Since $(x, g) \to g^{-1}f(gx)$ is a continuous map of $U \times W$ into V, it follows from the lemma that $F \mid U$ is continuous and hence F is continuous at x_0. Clearly $x \notin GS \Rightarrow f(gx) = 0$ for all $g \in G$ so $F(x) = 0$. Finally

$$\begin{aligned} F(\gamma x) &= \int g^{-1} f(g\gamma x)d\mu(g) \\ &= \int \gamma g^{-1}f(gx)\, d\mu(g) \\ &= \gamma \int g^{-1} f(gx)\, d\mu(g) = \gamma F(x) \end{aligned}$$

taking γ outside the integral sign being justified because $v \to \gamma v$ is a linear map of V. q.e.d.

1.2.7. THEOREM. *Let X be a proper G-space and v an element of a linear G-space V. A necessary and sufficient condition that there exist an equivariant map F of X into V with $F(x_0) = v$ is that $G_{x_0} \subseteqq G_v$.*

PROOF. Since $G_x \subseteqq G_{f(x)}$ is a trivial consequence of equivariance, necessity is clear. Now let $C_S(X)$ denote the space of continuous real valued functions on X which have small support. Because finite unions of small sets are small $C_S(X)$ is a linear space and clearly $T: f \to \int f(gx_0)g^{-1}vd\mu(g)$ is a linear map of $C_S(X)$ into V, so $W = T(C_S(V))$ is a linear subspace of V. If $Tf = v$ then, by 1.2.6, $F(x) = \int f(gx)g^{-1}vd\mu(g)$ will be an equivariant map of X into V with $F(x_0) = v$. Hence it will suffice to show that if $G_{x_0} \subseteqq G_v$ then $v \in W$. Since subspaces of a finite dimensional real vector space are always closed, it will in turn be sufficient to show that v is adherent to W. Let K be any convex neighborhood of v. Since G_x is compact and $G_xv \subseteqq K$ we can find a compact neighborhood U of G_x in G such that $U^{-1}v \subseteqq K$. By 1.1.6, there is a neighborhood S of x such that $gx \in S$ implies $g \in U$ and we can assume that S is small. Since X is completely regular we can find a continuous non-negative real valued function f with support in S such that $f(x_0) \neq 0$. By multiplying f by a positive constant we can assume that $\int_U f(gx_0)d\mu(g) = 1$. Since $g \notin U \Rightarrow gx_0 \notin S \Rightarrow f(gx_0) = 0$ it follows that $d\nu(g) = f(gx_0)d\mu(g)$ defines a positive Radon measure of mass one in G with support in U. Since $g^{-1}v \in K$ for $g \in U$, and K is convex it follows that $T(f) = \int f(gx_0)g^{-1}vd\mu(g) = \int g^{-1}vd\nu(g)$ is in K. Thus $K \cap W \neq \emptyset$. Since every neighborhood of v includes a convex neighborhood of v it follows that v is adherent to W. q.e.d.

1.2.8. PROPOSITION. *If X is a proper G-space then X/G is completely regular.*

PROOF. We already know that X/G is T_1, by 1.1.4. Let $\tilde{x}_0 = \Pi_X(x_0) \in X/G$ and let F be a closed subset of X/G not containing $\tilde{x}_0$. Let S be a small neighborhood of x_0 disjoint from $\Pi_X^{-1}(F)$ and let f be a non-negative real valued function on X with support in S such that $f(x_0) > 0$. By 1.2.6, $f^*(x) = \int f(gx)d\mu(g)$ is an invariant, continuous, real valued function on X with $f^*(x_0) > 0$ and having support in GS (and hence disjoint from $\Pi_X^{-1}(F)$). Since f^* is invariant $f = f^* \, \Pi_X^{-1}$ is a well-defined function on X/G which satisfies $f(\tilde{x}_0) > 0$ and $f \mid F \equiv 0$. Finally Π_X is an open map (if O is open in X then $\tilde{O} = \Pi_X(O)$ is open in X/G because $\Pi_X^{-1}(\tilde{O}) = \bigcup_{g\in G} gO$ is a union of open sets) which immediately implies that f is continuous. q.e.d.

COROLLARY 1. *A G-space X is proper if and only if X is a Cartan G-space and X/G is regular.*

PROOF. Necessity from 1.2.3 and 1.2.8 and sufficiency from 1.2.5.

COROLLARY 2. *If X is a Cartan G-space then X/G is locally completely regular.*

PROOF. 1.2.4 and 1.2.8.

Note that this is a considerable strengthening of 1.1.4 since a space that is even locally T_1 is T_1.

LEMMA. *If a G-space X has the property that every pair of its points has relatively thin neighborhoods then X/G is Hausdorff.*

PROOF. Let $R = \{(x, gx) \mid x \in X, g \in G\}$. We must show that R is closed in $X \times X$. Suppose $(x_\alpha, g_\alpha x_\alpha) \to (x, y)$ and let U and V be relatively thin neighborhoods of x and y respectively. We can suppose $x_\alpha \in U$ and $g_\alpha x_\alpha \in V$ so $g_\alpha \in ((U, V))$ and hence by passing to a subnet we can suppose that $g_\alpha \to g$. Since $x_\alpha \to x$ we have $y = \lim g_\alpha x_\alpha = gx$ so $(x, y) = (x, gx) \in R$. q.e.d.

1.2.9. THEOREM. *If X is a locally compact G-space then the following are equivalent.*

(1) *Given x, y in X there exist relatively thin neighborhoods U and V of x and y.*

(2) *X is a Cartan G-space and X/G is Hausdorff.*

(3) *X is a proper G-space.*

(4) *Every compact subset of X is small.*

(5) *Every compact subset of X is thin (or equivalently, if $K \subseteq X$ is compact then $((K, K))$ is compact).*

PROOF. Clearly (1) implies that X is Cartan and, by the lemma, that X/G is Hausdorff. Since X/G is locally compact (because, as was shown in the proof of 1.2.8, Π_X is open) Hausdorff implies regular so (2) implies (3). Since in a proper G-space compact sets are small, we see that (3) implies (4) and since a compact set is thin relative to a small set, (4) implies (5). Finally if (5) holds let U and V be compact neighborhoods of x and y respectively. Then $U \cup V$ is thin and *a fortiori* U is thin relative to V. q.e.d.

The notion of a proper G-space seems to have originated with A. Borel who defined the notion for locally compact G-spaces (unpublished) and took condition (5) above as his definition. In case G is discrete the notion coincides with the classical notion of "properly discontinuous" and it was this that led to the choice of name.

1.3 Operations on G-spaces

In this section we study a number of the standard ways of forming new transformation groups out of old and investigate the extent to which these constructions preserve the properties of being Cartan and proper.

1.3.1. PROPOSITION. *If X is a proper (respectively, Cartan) G-space, H a closed subgroup of G and Y an H-invariant subspace of X then Y is a proper (respectively, Cartan) H-space.*

PROOF. Trivial.

1.3.2. PROPOSITION. *If X is a proper G-space and N is a closed normal subgroup of G then X/N is a proper G/N-space.*

PROOF. Recall that G/N acts on X/N by $(gN)(Nx) = Ngx$. Since X/N is completely regular (1.2.8 and 1.3.1) it follows easily that this in fact makes X/N a G/N-space. Moreover $X/N\big/G/N$ is canonically homeomorphic to X/G and so, by 1.2.8, again is completely regular. Thus by 1.2.5, it will suffice to show that X/N is a Cartan G/N space. Given $\tilde{x} = Nx$ in X/N let U be a thin neighborhood of x in X. Then $\tilde{U} = \{Ny \mid y \in U\}$ is a neighborhood of $\tilde{x}$ in X/N (because the projection map $y \to Ny$ is open). Moreover if Π is the canonical map of G onto G/N then it is immediately checked that $\Pi((U, U)) = ((\tilde{U}, \tilde{U}))$, hence since $((U, U))$ is relatively compact in G, $((U, U))$ is relatively compact in G/N, so $\tilde{U}$ is a thin neighborhood of $\tilde{x}$. q.e.d.

1.3.3. PROPOSITION. *Let X and Y be G-spaces. If X is a Cartan G-space (respectively, proper G-space) then so is $X \times Y$.*

PROOF. Recall that G acts on $X \times Y$ by $g(x, y) = (gx, gy)$. It is then easy to verify that if U and V are subsets of X then $((U \times Y, V \times Y)) = ((U, V))$. It follows that if U and V are relatively thin then so are $U \times Y$ and $V \times Y$. In particular if U is a thin (small) neighborhood of $x \in X$ then for all $y \in Y$ and set $U \times Y$ is a thin (small) neighborhood of (x, y). q.e.d.

Now let Y be a G-space, $\tilde{X}$ a completely regular space and $\tilde{f} : \tilde{x} \to Y/G$ a map. We recall the definition of the induced G-space $\tilde{f}^{-1}(Y)$. As a space $\tilde{f}^{-1}(Y) = \{(x, y) \in X \times Y \mid \tilde{f}(x) = \Pi_Y(y)\}$ and the action of G is given by $g(x, y) = (x, gy)$. We note that the map $f : (x, y) \to y$ is an equivariant map of $\tilde{f}^{-1}(Y)$ into Y and that if Π is the orbit map of $\tilde{f}^{-1}(Y)$ onto its orbit space then $h : \Pi(x, y) \to x$ is a homeomorphism of $\tilde{f}^{-1}(Y)/G$ with X such that $\tilde{f} \circ h \circ \Pi = \Pi_Y \circ f$. Moreover $\tilde{f}^{-1}(Y)$ can be characterized as a G-space (to within equivalence) by the existence of the maps h and f with these properties. This construction plays an important role in the classification theory of G-spaces. We now prove

1.3.4. PROPOSITION. *If Y is a proper G-space and $\tilde{f}$ a map of a completely regular space X into Y/G then $\tilde{f}^{-1}(Y)$ is a proper G-space.*

PROOF. Since $\tilde{f}^{-1}(Y)/G$ is homeomorphic to X and hence is completely regular it will suffice to show that $\tilde{f}^{-1}(Y)$ is a Cartan G-space, by 1.2.5. Let f be the natural equivariant map of $\tilde{f}^{-1}(Y)$ into Y, i.e., $(x, y) \to y$. It will suffice to show that if U is a thin subset of Y then $\tilde{f}^{-1}(U)$ is a thin subset of $\tilde{f}^{-1}(Y)$. Let $g \in ((f^{-1}(U), f^{-1}(U)))$. Then $g(x, y) = (x', y')$ where $y = f(x, y)$ and $y' = f(x', y')$ are in U. But $g(x, y) = (x, gy)$ hence $y' = gy$ and $g \in ((U, U))$. Thus $((f^{-1}(U), f^{-1}(U)))$ is a subset of $((U, U))$ and so is relatively compact in G. q.e.d.

2.1 Slices and kernels

Henceforth we assume that G is not only locally compact but also a Lie group. If H is a closed subgroup of G then by a *local cross section in G/H* we shall mean a differentiable map $\chi : U \to G$ where U is an open neighborhood of H in G/H, $\chi(H) = e$ and $\chi(\gamma) \in \gamma$ for $\gamma \in U$. The existence of local cross-sections is well known (see for example [1, page 109]).

2.1.1. DEFINITION. Let X be a G-space and H a closed subgroup of G. A subset S of X will be called an *H-kernel* (over $\Pi_X(S)$) if there exists an equivariant map $f : GS \to G/H$ such that $f^{-1}(H) = S$. If in addition GS is open in X we call S an *H-slice* in X. If $GS = X$ we call S a *global H-slice for X*. If $x \in X$ then by a *slice at x* we mean a G_x-slice in X which contains x.

Suppose S is an H-kernel in X and let $f : GS \to G/H$ be an equivariant map with $S = f^{-1}(H)$. Then if $y \in GS$ we have $y = gs$ for some $g \in G$ and $s \in S$ so $f(g) = f(gs) = gf(s) = gH$. This shows that f is uniquely determined by S and we denote it by f^S. Clearly then if $\tilde{U}$ is a subset of X/G and $U = \Pi_X^{-1}(\tilde{U})$ then $f \to f^{-1}(H)$ and $S \to f^S$ are mutually inverse one-to-one correspondences between the set of equivariant maps of U into G/H and the set of H-kernels over $\tilde{U}$.

If S is an H-kernel in the G-space X then S is clearly an H-invariant subset of X and so an H-space. If $s \in S$ then of course $H_s = G_s \cap H$. On the other hand if $g \in G_s$ then $gH = gf^S(s) = f^S(gs) = f^S(s) = H$ so $g \in H$, i.e., $G_s \subseteq H$ and so $H_s = G_s$.

2.1.2. PROPOSITION. *Let S be an H-kernel in the G-space X and let $\chi : U \to G$ be a local cross-section in G/H. Then if $g_0 \in G$ the map*

$F:(u, s) \to g_0\chi(g_0^{-1}u)s$ is a homeomorphism of $g_0U \times S$ onto an open neighborhood of g_0S in GS. Moreover $f^S(F(u, s)) \equiv u$.

PROOF. Recall that $f^S(gx) = gH$ for $g \in G$, $s \in S$, hence since $\chi(g_0^{-1}u)H = g_0^{-1}u$ (by definition of a local cross-section) it follows that $f^S(F(u, s)) = g_0\chi(g_0^{-1}u)H = g_0g_0^{-1}u = u$. Thus $F(g_0U \times S) = f^{S-1}(g_0U)$ is an open neighborhood of g_0S in GS. Since the continuity of F is clear we complete the proof by showing that if $F(u_\alpha, s_\alpha) \to F(u, s)$ then $u_\alpha \to u$ and $s_\alpha \to s$. In fact $u_\alpha = f(F(u_\alpha, s_\alpha)) \to f(F(u, s)) = u$. Then $\chi(g_0^{-1}u_\alpha)^{-1} \to \chi(g_0^{-1}u)^{-1}$ and since $\chi(g_0^{-1}u_\alpha)s^\alpha = g_0^{-1}F(u_\alpha, s_\alpha) \to g_0^{-1}F(u, s) = \chi(g_0^{-1}u)s$ it follows that $s_\alpha \to s$. q.e.d.

COROLLARY. *If S is an H-kernel in X and W is open in S then GW is open in GS.*

PROOF. Taking $g_0 = e$ in the theorem we see that $F(U \times W)$ is open in GS and hence that $GF(U \times W)$ is open in GS. But clearly $GF(U \times W) = GW$. q.e.d.

2.1.3. PROPOSITION. *Let S_1 and S_2 be H-kernels in G-spaces X_1 and X_2 respectively and let f_0 be an H-equivariant map of S_1 into S_2. Then there is a unique G-equivariant map f of GS_1 onto GS_2 such that $f \mid S_1 = f_0$; namely $f(gs) = gf_0(s)$ for $g \in G$, $s \in S$. Moreover if f_0 imbeds S_1 in S_2 then f imbeds GS_1 in GS_2.*

PROOF. It is clear that if f exists it is given by $f(gs) = gf_0(s)$. Since f_0 is H-equivariant it follows that $H_s \subseteq H_{f_0(s)}$ and so that $G_s \subseteq G_{f_0(s)}$ from which it follows that the above formula gives a well-defined function f from GS_1 into GS_2 which is clearly equivariant. It remains only to check the continuity of f, the continuity of f^{-1} when f_0^{-1} exists and is continuous then following by symmetry. Let $\chi : U \to G$ be a local cross section in G/H. By 2.1.2, the map $F_i : (u, s) \to g_0\chi(g_0^{-1}u)s$ is a homeomorphism of $g_0U \times S_i$ onto a neighborhood of g_0S_i in GS_i for $i = 1,2$ and $g_0 \in G$. Since $fF_1(u, s) = f(g_0\chi(g_0^{-1}u)s) = g_0\chi(g_0^{-1}u)f_0(s) = F_2(u, f_0(s))$ the continuity of f on g_0S_1 is clear and since g_0 is arbitrary, f is continuous on GS. q.e.d.

If S is any H-space, H being a closed subgroup of G, then it is easy to construct a G-space X in which S occurs as an H-slice. Namely X is the fiber bundle with fiber S associated with the principal H-bundle G over G/H (the action being $h \circ g = gh^{-1}$ of course). For details see [5] or [10, 1.7.14]. From this and 2.1.3, we have

THEOREM. *If G is a Lie group, H a closed subgroup and S an H-space then there exists a G-space X having S as a global H-slice. Moreover X is essentially unique in the sense that if X' is another G-space in which S occurs as a global H-slice then the identity map of S extends uniquely* *to an equivariant homeomorphism of X onto X'.*

We now come to an important intrinsic characterization of an H-kernel.

2.1.4. THEOREM. *Let X be a G-space and H a closed subgroup of G. If S is an H-kernel in X then:*
(1) *S is closed in GS;*
(2) *S is invariant under H;*
(3) $gS \cap S \neq \emptyset \Rightarrow g \in H$, i.e., $((S, S)) = H$.
If H is compact then in addition
(4) *S has a thin neighborhood in GS.*
Conversely if conditions (1) - (4) *hold then H is compact and S is an H-kernel in X.*

PROOF. If S is an H-kernel in X then it is immediately that (1) - (3) hold. If H is compact let $\tilde{U}$ be a compact neighborhood of H in G/H so that $U =$ union of cosets in $\tilde{U}$ is a compact neighborhood of e in G. Then $W = f^{S^{-1}}(\tilde{U})$ is a neighborhood of S in GS and it is easily checked that $((W, W)) \subseteqq U$ so that W is thin and (4) holds.

Conversely, supposing (1) - (4) hold, (2) and (4) show immediately that H is compact. If $g_1, g_2 \in G$, $s_1, s_2 \in S$ and $g_1 s_1 = g_2 s_2$ then $g_2^{-1} g_1 s_1 = s_2$ so by (3) $g_2^{-1} g_1 \in H$ and $g_1 H = g_2 H$, hence the function $f: gs \to gH$ of GS onto G/H is well defined. Clearly f is equivariant and $S = f^{-1}(H)$ so it remains only to show that f is continuous. Let $g_\alpha s_\alpha \to gs$. We must show that $g_\alpha H \to gH$. Since $g^{-1} g_\alpha s_\alpha \to s$ and since $g^{-1} g_\alpha H \to H$ implies $g_\alpha H \to H$ we can suppose $g = e$. Now if $g_\alpha H \not\to H$ there is a neighborhood U of H in G such that $g_\alpha \notin U$ for arbitrarily large α, hence by passing to a subnet we can suppose that no subnet of g_α converges to a point of H. On the other hand if we let V be a thin neighborhood of S in GS then $g_\alpha s_\alpha \in V$ for α sufficiently large so $g_\alpha \in ((V, V))$ and hence some subnet $g_{\alpha(\beta)}$ of g_α converges to $g \in G$. We will show that $g \in H$, a contradiction which will complete the proof. In fact since $g_{\alpha(\beta)} s_{\alpha(\beta)} \to s$ it follows that $s_{\alpha(\beta)} \to g^{-1}s$. Since S is closed in GS it follows that $g^{-1}s \in S$ and so by (3) g^{-1} (and hence g) is in H. q.e.d.

Of course if G is compact then condition (4) of 2.1.4 is automatically satisfied. However in the general case the following example shows that (4) is not a consequence of (1) - (3). Take $G = R$, $X = R \times R$, $H = e$ and define $t(x, y) = (x, y + t)$ (vertical translation) and let $S = \{(x, 1/x^2) \mid x \neq 0\} \cup (0, 0)$.

2.1.5. PROPOSITION. *Let H and K be closed subgroups of the Lie group G with $H \subseteqq K$. Let X be a G-space, T a K-kernel (respectively, K-slice) in X and S an H-kernel (respectively, H-slice) in the K-space T. Then*

S is an H-kernel (respectively, H-slice) in X.

PROOF. If $gS \cap S \neq \emptyset$ then *a fortiori* $gT \cap T \neq \emptyset$ so $g \in K$ and, since T is a K-kernel, $g \in H$. It follows as in the proof of 2.1.4 that the function $f: gs \to gH$ of GS into G/H is well defined and equivariant, and clearly $S = f^{-1}(H)$. We will now show that f is continuous which shows that S is an H-kernel. As in 2.1.4, it suffices to show that if $g_\alpha s_\alpha \to s$ then $g_\alpha H \to H$. Since T is a K-kernel and $S \subseteqq T$ $g_\alpha K \to K$. Let χ: $U \to G$ be a local cross-section in G/K. We can assume that $g_\alpha K \in U$. Put $\gamma_\alpha = \chi(g_\alpha k_\alpha)$. Then $\gamma_\alpha \to e$ and $g_\alpha = \gamma_\alpha k_\alpha$ where $k_\alpha \in K$. Also $\lim k_\alpha s_\alpha = \lim \gamma_\alpha^{-1} \lim g_\alpha s_\alpha = s$ and since S is an H-kernel in the K-space T it follows that $k_\alpha H \to H$. But then $g_\alpha H = \gamma_\alpha k_\alpha H \to H$ also. If S is an H-slice in T then KS is open in T and, by the corollary of 2.1.2, $GS = GKS$ is open in GT. If also T is a K-slice in X then GT is open in X so GS is open in X hence S is an H-slice in X. q.e.d.

2.1.6. DEFINITION. A subset S of a G-space X will be called a *near-slice at x* if $x \in S$, $G_x S = S$ and there exists a local cross-section χ: $U \to G$ in G/G_x such that $(u, s) \to \chi(u)s$ is a homeomorphism of $U \times S$ onto an open neighborhood of x in X.

Note that by 2.1.2 a slice at x is a near slice at x. The concept of a near slice will play only a transitory technical role. We are really interested in showing that in a Cartan G-space there is always a slice at each point. Near slices however are easier to come by and the next result says it will be enough to show their existence.

2.1.7. PROPOSITION. *If x is a Cartan G-space and S^* is a near slice at $x \in X$ then there is a neighborhood S of x in S^* which is a slice at x.*

PROOF. Let $\chi: U \to G$ be a local cross-section in G/G_x such that $(u, s) \to \chi(u)s$ is a homeomorphism of $U \times S^*$ onto an open neighborhood of x. By 1.1.6 we can choose an open neighborhood V of x such that $((V, V)) \subseteqq U'$ where U' is the union of the cosets in U. Since G_x is compact we can suppose that V is invariant under G_x and hence that $S = S^* \cap V$ is invariant under G_x. Since S is open in S^* it follows that $U'S = \{\chi(u)s \mid u \in U, s \in S\}$ is open in X and hence that $GS = GU'S$ is open X. If $gS \cap S \neq \emptyset$ then $g \in U'$ so $gG_x \in U$. Choosing $s_1 \in S$ so that $gs_1 = s_2 \in S$ and putting $\chi(gG_x) = gh$ we see that $\chi(gG_x)h^{-1}s_1 = gs_1 = s_2 = \chi(G_x)s_2$. Now $h \in G_x$ and S is G_x-invariant so $h^{-1}s_1 \in S$ and since $(u, s) \to \chi(u)s$ is a homeomorphism on $U \times S$ it follows that $gG_x = G_x$, i.e., $g \in G_x$. Thus $((S, S)) = G_x$ and it follows as in 2.1.4 that $gs \to gH$ gives a well-defined equivariant function from GS into G/G_x with $S = f^{-1}(G_x)$. It remains to show that $g_\alpha s_\alpha \to s$ implies $g_\alpha G_x \to G_x$. Now $g_\alpha s_\alpha$ is eventually in V so $g_\alpha G_x$

is eventually in U. Putting $g_\alpha = \chi(g_\alpha G_x)h_\alpha$ for large α we have $h_\alpha \in G_x$ so $h_\alpha s_\alpha \in S$. Since $\chi(g_\alpha G_x)h_\alpha s_\alpha \to \chi(G_x)s$ it follows (using again that $(u,s) \to \chi(u)s$ is a homeomorphism on $U \times S$) that $g_\alpha G_x \to G_x$. q.e.d.

2.1.8. PROPOSITION. *Let X and Y be G-spaces and $f: X \to Y$ an equivariant map. Let $x_0 \in X$ and suppose $G_{x_0} = G_{f(x_0)}$. Then if S^* is a near slice at $f(x_0)$ in Y, $S = f^{-1}(S^*)$ is a near slice at x_0 in X.*

PROOF. Put $G_{x_0} = G_{f(x_0)} = H$ and let $\chi : U \to G$ be a local cross-section in G/H such that $F^* : (u, s^*) \to \chi(u)s^*$ is a homeomorphism of $U \times S^*$ onto an open neighborhood O^* of $f(x_0)$ in Y. Then $O = f^{-1}(O^*)$ is an open neighborhood of x_0 in X and to complete the proof we will show that $F: (u, s) \to \chi(u)s$ is a homeomorphism of $U \times S$ onto O. The continuity of F is clear and since $fF(u, s) = F^*(u, f(s))$ it follows that $F(U \times S) \subseteqq O$. If $x \in O$ then $f(x) = \chi(u)s^*$ so $f(\chi(u)^{-1}s) = s^*$ and $\chi(u)^{-1}x \in S$. Since $F(u, \chi(u)^{-1}x) = x$ it follows that $F(U \times S) = O$. To complete the proof we show that $F(u_\alpha, s_\alpha) \to F(u, s) \Rightarrow u_\alpha \to u$ and $s_\alpha \to s$ which implies that F is one-to-one and that F^{-1} is continuous. In fact $F^*(u_\alpha, f(s_\alpha)) = fF(u_\alpha, s_\alpha) \to fF(u, s) = F^*(u, f(s))$ and since F^* is a homeomorphism $u_\alpha \to u$. That $s_\alpha \to s$ now follows just as in the proof of 2.1.2. q.e.d.

2.2. Slices and near slices in differentiable G-spaces

LEMMA. *Let X be a differentiable G-space, $x \in X$ and S^* a submanifold of X containing x and invariant under G_x. Denote by S^*_x, $(Gx)_x$, and X_x the tangent spaces at x to S^*, Gx, and X respectively and suppose that S^*_x is a linear complement to $(Gx)_x$ in X_x. Then there exists a local cross-section $\chi : U \to G$ in G/G_x and a G_x-invariant open submanifold S of S^* containing x such that $(u, s) \to \chi(u)s$ is a diffeomorphism of $U \times S$ onto an open neighborhood of x. Hence S is a near slice at x.*

PROOF. Let $\chi^* : U \to G$ be a local cross section in G/G_x and define $F^* : U^* \times S^* \to X$ by $F(u, s) = \chi^*(u)s$. Clearly F is differentiable and by a general fact about differentiable G-spaces (which is an immediate consequence of the uniqueness theorem for solutions of ordinary differential equations (see for example [10, 1.1.22]) δF^* maps the tangent space to $U^* \times \{x\}$ at (G_x, x) isomorphically onto $(Gx)_x$. On the other hand it is clear from $F(G_x, s) \equiv s$ that δF^* maps the tangent space to $G_x \times S^*$ at (G_x, x) isomorphically onto S^*_x, and hence δF^* is an isomorphism of the tangent space to $U^* \times S^*$ at (G_x, x) onto X_x, and by the implicit function theorem there exists a neighborhood U of G_x in U^* and a neighborhood S' of x in S^* such that $F^* \mid U \times S'$ is a diffeomorphism onto a neighborhood of x in X. Since G_x is compact we can find an open neighborhood S of x included in S' which is invariant under G_x. Taking $\chi = \chi^* \mid U$ the theorem is proved. q.e.d.

2.2.1. PROPOSITION. (Koszul [5, p. 139]). *If X is a differentiable G-space, $x \in X$ and G_x is compact, then there exists a near-slice at x in X.*

PROOF. Choose a Riemannian metric for X invariant under G_x. Let $r > 0$ be chosen so small that there is a Riemannian normal coordinate system at x of radius r and let S^* be the union of geodesic segments of length r starting from x in a direction orthogonal to G_x. Clearly S^* satisfies the condition of the lemma. q.e.d.

2.2.2. PROPOSITION. *If X is a Cartan differentiable G-space and $x \in X$ then there exists a slice at x.*

PROOF. Immediate from 2.2.1 and 2.1.7.

2.2.3. REMARK. If X is a Cartan differentiable G-space, $x \in X$ and S is a slice at x constructed by the above method, then GS is a tubular neighborhood of Gx relative to a G_x invariant Riemannian metric for X and the equivariant map $f^S : GS \to G/G_x$ is just the usual fiber map when G/G_x is identified with Gx by $gG_x \to gx$. Hence in particular f^S is differentiable. Moreover it follows that if $\chi : U \to G$ is a local cross-section G/G_x and $g_0 \in G$ then the map $F : (u, s) \to g_0\chi(g_0^{-1}u)s$ is a diffeomorphism of $g_0U \times S$ onto an open neighborhood of g_0S in GS.

2.3. Existence of slices in a Cartan G-space

We shall say that a Lie group is *of type* S if there is a slice at each point of every proper G-space. Our next goal is to prove

2.3.1. PROPOSITION. *Every Lie group is of type S.*

We will arrive at the full proof of 2.3.1 by first proving a number of special cases. By a matrix group we mean a Lie group G which admits a faithful continuous linear representation in a finite dimensional real vector space. By a theorem in the next section of this paper (3.2), if G is a matrix group, X a proper G-space, and $x \in X$ there is a linear G-space V and a $v \in V$ such that $G_x = G_v$. By 2.2.1, there is a near slice S^* at v relative to V, and by 1.2.7, there is an equivariant map $f : X \to V$ with $f(x) = v$. By 2.1.8, $f^{-1}(S^*)$ is a near slice at x relative to X and so by 2.1.7 (and 1.2.3) there is a slice at x in X. This proves what is perhaps the crucial special case of 2.3.1, namely:

Case 1. A matrix group is of type S.

Case 2. If a Lie group G has a closed normal subgroup N such that G/N is of type S and such that KN is of type S for each compact subgroup K of G then G is of type S.

PROOF. Let X be a proper G-space and let $x \in X$. Let Π denote the orbit map of X onto X/N. Now X/N is a proper G/N-space by 1.3.2, hence there exists a slice $\tilde{T}$ at $\tilde{x} = \Pi(x)$ relative to X/N. We claim $T = \Pi^{-1}(\tilde{T})$ is a G_xN slice in X. In fact recalling that G/N acts on X/N by $(gN)\,\Pi(y) = \Pi(gy)$ it is clear in the first place that $GT = \Pi^{-1}(G/N\,\tilde{T})$ and hence, since $GN\tilde{T}$ is open in X/N, GT is open in X. Next if h is the canonical identification of $G/N\big/G_xN/N$ with G/G_xN then $h \circ f^{\tilde{T}} \circ \Pi = f^T$ is easily seen to be an equivariant map of GT onto G/G_xN with $f^{T-1}(G_xN) = T$.

Now T is a proper G_xN-space by 1.3.1, hence we can find a slice S at x in the G_xN-space T. Since the isotropy group at x in T (considered as a G_xN-space) is G_x (see remark preceding 2.1.2) it follows that S is a G_x-slice in the G_xN-space T and by 2.1.5, that S is a slice at x in X. q.e.d.

Case 3. A discrete group is of type S.

PROOF. Let G be a discrete group, X a proper G-space and $x \in X$. Let W be a small neighborhood of x. Since G_x is compact we can suppose that W is invariant under G_x. It is then trivial from the definition that W is a near slice at x and hence by 2.1.7 that some smaller neighborhood of x is a slice at x. q.e.d.

In what follows G_0 will as usual denote the identity component of a Lie group G.

Case 4. If G is a Lie group such that KG_0 is of type S for each compact subgroup K of G then G is of type S.

PROOF. Immediate from cases 2 and 3.

Case 5. If G_0 is compact then G is of type S.

PROOF. If K is a compact subgroup of G then KG_0 is a compact subgroup of G. Since all compact Lie groups are matrix groups and hence of type S this case follows from case 4.

Case 6. A Lie group G which is an extension of a discrete normal subgroup N by a group of type S is itself of type S.

PROOF. Let K be a compact subgroup of G. Since G/N is by hypothesis of type S it will suffice by case 2 to show that KN is of type S. For this it will be enough to show that K is open in KN, for then $(KN)_0 = K_0$ is compact and by case 5, KN is of type S. Choose a neighborhood V of K such that $V \cap N \subseteqq K$ (N is discrete). Since K is compact we can assume V is a union of right K cosets. Then if $k \in K$, $n \in N$, and $kn \in V$ it follows that $Kn \subseteqq V$ so $n \in V \cap N \subseteqq K$, hence $kn \in K$. This shows $KN \cap V = K$ so K is open in KN. q.e.d.

Case 7. A Lie group G which is an extension of a normal vector subgroup V by a compact group is a matrix group and hence of type S.

PROOF. By a theorem of Iwasawa [4, Lemma 3.7] there is a compact subgroup H of G complimentary to V (i.e., $HV = G$, $H \cap V = e$, so G is the semidirect product of H and V). Let A be the adjoint representation of H in V and R' any faithful representation of H in a finite dimensional real vector space. Then $R = R' \oplus A$ is a faithful representation of H in a finite dimensional real vector space W and V is an invariant subspace of W such that $R \,/\, V \cong A$. It is clear then that the group of affine transformations of W generated by the image of R and the translations $T_v : w \to w + v$ where $v \in V$ is isomorphic to G. Since the full affine group of W is a matrix group (it is its own adjoint group) *a fortiori* G is a matrix group. q.e.d.

Case 8. If G is a Lie group, C the centralizer of G_0 and K a compact subgroup of G then $\Gamma = KC_0$ is a matrix group and hence of type S.

PROOF. By case 7 it will suffice to find a vector group V which is a closed normal subgroup of Γ such that Γ/V is compact. The kernel N of the adjoint representation of Γ on its Lie algebra clearly includes C_0, hence Γ/N is compact and so the adjoint representation of Γ is completely reducible. The subgroup T of C_0 generated by one parameter subgroups with compact closure is a torus which is characteristic in C_0 and hence normal in Γ. Since the Lie algebras c and t of C_0 and T are ideals in the Lie algebra γ of Γ it follows from the complete reducibility of ad (Γ) that there is an ideal v of γ complimentary to t in c. The normal subgroup V of Γ generated by v is clearly a vector group closed in C_0 and hence in Γ. Since $C_0/V \cong T$ is compact and $\Gamma/V \big/ C_0/V \cong \Gamma/C_0 \cong K/K \cap C_0$ is compact, it follows that Γ/V is compact. q.e.d.

PROOF OF 2.3.1. Let C be the centralizer of G_0. Then G/C is a matrix group (C is the kernel of the adjoint representation of G on its Lie algebra) and hence of type S. Since $G/C_0 \big/ C/C_0 \cong G/C_0$ and C/C_0 is a discrete normal subgroup of G/C_0 it follows from case 6 that G/C_0 is of type S. But if K is any compact subgroup of G then KC_0 is of type S by case 8. Hence by case 2, G is of type S. q.e.d.

2.3.2. THEOREM. *Let G be a Lie group, X a G-space and $x \in X$. The following two conditions are equivalent:*

(1) *G_x is compact and there is a slice at x.*

(2) *There is a neighborhood V of x in X such that $\{g \in G \mid gV \cap V \neq \varnothing\}$ has compact closure in G.*

PROOF. That (1) implies (2) follows directly from 2.1.4. Conversely suppose that (2) holds, i.e., that x has a thin neighborhood V. Then by 1.2.4, GV is a proper G-space and hence G_x is compact and by 2.3.1 there is a slice S at x in GV. Since GV is open in X, S is also a slice at x in X. q.e.d.

2.3.3. THEOREM. *Let G be a Lie group and X a G-space. Then the following two conditions are equivalent.*

(1) *For each $x \in X$, G_x is compact and there is a slice at x.*

(2) *X is a Cartan G-space.*

PROOF. An immediate corollary of 2.3.2 and definition 1.1.2.

COROLLARY 1. *If G is a Lie group and X a Cartan G-space then every orbit of X is an equivariant retract of an invariant neighborhood of itself.*

PROOF. Given $x \in X$ let S be a slice at x and let $h : G/G_x \to Gx$ be the equivariant homeomorphism (1.1.5), $gG_x \to gx$. Then $h \circ f^S$ is an equivariant map of the invariant neighborhood GS of Gx onto Gx. Moreover if $y = gx \in Gx$ then $f^S(y) = gG_x$ so $h \circ f^S(y) = y$ and $h \circ f^S$ is a retraction of GS onto Gx.

COROLLARY 2. *Let G be a Lie group and X a Cartan G-space. If $x \in X$ then there is a neighborhood V of x such that $y \in V$ implies G_y is conjugate in G to a subgroup of G_x. Moreover if O is any neighborhood of e in G then V can be chosen so that in fact for each $y \in V$ we have $G_y \subseteqq gG_xg^{-1}$ for some $g \in O$.*

PROOF. Let S be a slice at x, put $f = f^S$ and $V = f^{-1}(OG_x)$. Then V is a neighborhood of x in GS and hence in X. Since f is equivariant, if $y \in V$ then $G_y \subseteqq G_{f(y)}$. Now $f(y) = gG_x$ for some $g \in O$ and the isotropy group of gG_x in G/G_x is clearly gG_xg^{-1}. q.e.d.

COROLLARY 3. *Let G be a Lie group, X a Cartan G-space and $x \in X$. If H is any closed subgroup of G which includes G_x then there exists an H-slice in X containing x.*

PROOF. Let S be a slice at x and let $f : G/G_x \to G/H$ be the canonical equivariant map, i.e., $f(gG_x) = gH$. Then $f \circ f^S$ is an equivariant map of GS into G/H so $(f \circ f^{S^{-1}}) = HS$ is an H-slice containing x. q.e.d.

3.1. Extending a representation

Let G be a Lie group and V a linear G-space. By a matrix element of V we mean a real valued function on G of the form $g \to l(gv)$ where $v \in V$ and $l \in V^*$, the space of linear functionals on V. Since we can form direct sums and Kronecker products of representations of G it follows that the set $R(G)$ of real valued functions on G which are matrix elements of some linear G-space form a subalgebra of the algebra of all continuous real valued functions on G. If H is a compact subgroup of G we will write $R(G) \mid H$ for the algebra of real valued functions on H which are restrictions of elements in $R(G)$. Now let U be an irreducible linear H-space and suppose V is a linear G-space which, considered as a linear H-space by restriction, contains no invariant subspace equivalent to U. Then it is an immediate consequence of the "Schur orthogonality relations" that if f is a matrix element of V and k a matrix element of U then $f \mid H$ is orthogonal to k with respect to Haar measure on H. Thus if there were no linear G-space V which considered as an H-space by restriction had an invariant subspace equivalent to U, then every matrix element of U would be orthogonal to $R(G) \mid H$. But now suppose G is a matrix group. Then clearly $R(G)$ separates points of G so *a fortiori* $R(G) \mid H$ separates points of H and by the Stone-Weierstrass theorem $R(G)/H$ is uniformly dense in the space of continuous functions on H. It follows that no non-zero continuous function on H is orthogonal to $R(G) \mid H$. Hence we have proved that if U is an irreducible linear H-space there is always a linear G-space V which, considered as an H-space by restriction, contains U as an invariant linear subspace. Now since every linear H-space is a direct sum of irreducible ones, the same result clearly holds even when U is not irreducible. That is we have proved

THEOREM. *If G is a matrix group, H a compact subgroup, and U a linear H-space, then there is a linear G-space V which, considered as a linear H-space by restriction, contains U as an invariant linear subspace.*

3.2. A characterization of matrix groups

THEOREM. *A necessary and sufficient condition for a Lie group G to be a matrix group is that given a compact subgroup H of G there exists a linear G-space V and a $v \in V$ such that $G_v = H$.*

PROOF. The condition is clearly sufficient for taking $H = \{e\}$ the corresponding V is an effective linear G-space so by definition G is a matrix group.

Conversely suppose G is a matrix group. Then there exists a continuous one-to-one representation f of G in the group $\mathrm{GL}(n, R)$ of non-singular $n \times n$ real matrices and since H is compact we can assume that $H^* = f(H) \subseteqq \mathrm{O}(n)$ the group of orthogonal $n \times n$ matrices. By [9, Lemma a] there is a linear $\mathrm{O}(n)$ space U with a $u \in U$ such that $\mathrm{O}(n)_u = H^*$. By 3.1 there is a linear $\mathrm{GL}(n, R)$ space U' such that, considered as an $\mathrm{O}(n)$ space by restriction, U is an invariant linear subspace. Then clearly $H^* =$

$O(n) \cap GL(n, R)_u$. On the other hand $GL(n, R)$ acts linearly on the space W of quadratic forms of dimension n so that $O(n)$ is the isotropy group of the unit quadratic form w. Then $V = W \oplus U'$ is a linear $GL(n, R)$ space and $GL(n, R)_{(w, u)} = GL(n, R)_w \cap GL(n, R)_u = O(n) \cap GL(n, R) = H^*$. Now V becomes a G-space by $gv = f(g)v$ and clearly $G_{(w, u)} = f^{-1}(H^*) = H$. q.e.d.

4.1. Local triviality of principal bundles

The results of this section are not new. For the case of compact Lie groups they are due to Gleason [2]. The general case has not, to the author's knowledge, appeared in published form but will be found in mimeographed notes of a Bourbaki seminar lecture given by J.-P. Serre on March 27, 1950 (*Extensions des groupes localement compact*).

THEOREM. *If G is a Lie group then a G-principal bundle is locally trivial if and only if it is a Cartan principal bundle.*

PROOF. In a principal G-bundle a slice at x is clearly the same as a local cross-section at x. Since local triviality is equivalent to the existence of a local cross-section through each point, the theorem is an immediate consequence of 1.1.3 and 2.3.3. q.e.d.

Now let X be an arbitrary topological group and G a closed Lie subgroup of X. Then X is a G-principal bundle under the action $g \circ x = xg^{-1}$, the orbits being the left G-cosets. If R is the set of pairs in $X \times X$ belonging to the same orbit, then the map $f: R \to G$ such that $x_2 = f(x_1, x_2)x_1$ is given by $f(x_1 x_2) = x_2^{-1}x_1$, which is continuous, hence X is a Cartan principal bundle and

COROLLARY. *If X is a topological group and G is a closed Lie subgroup of X then the fibering of X by left G-cosets is locally trivial.*

4.2. The Montgomery-Zippin neighboring subgroups theorem

The theorem of this section is also not new and will be found in Montgomery and Zippin [7]. Their proof involves an ingenious application of Riemannian geometry. G. D. Mostow has given a proof for the case G compact within the usual circle of ideas of transformations groups in [8, Cor. 3.2]. Here we will show how the results of this paper allow us to extend Mostow's proof of the general case.

THEOREM (Montgomery-Zippin). *Let G be a Lie group, U a neighborhood of the identity in G and H a compact subgroup of G. There is a neighborhood of H in G such that any subgroup of G included in this neighborhood is conjugate to a subgroup of H by an element of U.*

PROOF. Let X be the space of compact non-empty subsets of G with the Hausdorff topology (i.e., a basic open set N of X is defined by choosing open sets $O_1, \cdots, O_n$ in G and letting $N = \{K \in X \mid K \cap O_i \neq \emptyset$ and $K \subseteqq O_1 \cup \cdots \cup O_n)$, alternatively metrize X as in corollary 3.2 of [8]. Then if $g \in G$ and $K \in X$ we define gK as usual to be $\{gk \mid k \in K\}$ and it is easily seen that X is a G-space. If H is a compact subgroup of G then $G_H = H$ and if U is a neighborhood of H in G then $\tilde{U} = \{K \in X \mid K \subseteqq U\}$ is a neighborhood of H in X. Thus 4.2.1 will be an immediate consequence of 2.3.3 once it is shown that X is a Cartan G-space. Let F be a non-empty compact subset of G and K a compact neighborhood of F in G. We shall show that the neighborhood $U = \{S \in X \mid S \subseteqq K\}$ of F in X is thin. In fact suppose $g \in ((U, U))$. Then there exists $S \in U$ such that $gS \in U$. Choose $s \in S$ (recall elements of X are non empty). Then $s \in K$ and $gs \in K$ so $g \in KK^{-1}$. Thus $(U, U) \subseteqq KK^{-1}$ which is compact. q.e.d.

4.3 Metrization of proper G-spaces

4.3.1. THEOREM. *Let G be a Lie group and X a proper differentiable G-space. Then X admits an invariant Riemannian metric.*

PROOF. For each $x \in X$ we can find a slice S_x at x which is "differentiable" in the sense of satisfying the properties mentioned in 2.2.3. Since X/G is locally compact, σ-compact and by 1.2.8 Hausdorff, we can choose a sequence $\{x_n\}$ in X such that $\Pi_X(S_{x_n})$ is a locally finite covering of X/G and hence GS_{x_n} is a locally finite covering of X. Since X/G is normal it is easily seen that we can find a neighborhood K_n of X_n in S_{x_n} such that K_n has compact closure in S_{x_n} and $\Pi_X(K_n)$ is a covering of X/G. Let f_n be a non-negative, differentiable, real valued function of S_{x_n} positive on K_n and with support a compact subset of S_{x_n}. Since G_{x_n} is compact we can suppose f_n is invariant under G_{x_n}. Then $f_n(gs) = f_n(s)\,(g \in G,\ s \in S_{x_n}) f_n(x) = 0,\ x \notin GS_n$ is a well-defined invariant differentiable function on S. Let V_n be the restriction of the tangent bundle of X to S_{x_n}. Then G_{x_n} acts naturally on V_n and since G_x is compact we can find an invariant Riemannian metric $\tilde{\gamma}_n$ on the vector bundle V_n. If u and v are vectors at a point gs of GS_n define $\gamma_n(u, v) = \tilde{\gamma}_n(\delta g^{-1}u, \delta g^{-1}v)$. Then γ_n is a well-defined invariant Riemannian metric for GS_{x_n} and $\sum_n f_n \gamma_n$ is an invariant Riemannian metric for X. q.e.d.

4.3.2. DEFINITION. X is a *Hilbert* G-space if X is a real Hilbert space and each operation of G on X is an orthogonal linear transformation.

LEMMA 1. *If G is a Lie group and H a compact subgroup then every Hilbert H-space X is an H-invariant closed linear subspace of some*

Hilbert G-space.

PROOF. It suffices to prove the theorem for each irreducible component of a direct sum decomposition of X, so we can assume that H acts irreducibly on X. Let f be some non-zero matrix element of the representation of H on X, e.g., $f(h) = (hx, x)$ for some non-zero $x \in X$. It follows from the Schur orthogonality relations that if V is a Hilbert G-space, u and v orthonormal elements of V and $m(g) = (gu, v)$ satisfies $\int f(h)m(h)dh \neq 0$, then when V is considered an H-space by restriction it contains a subrepresentation equivalent to X. It will suffice then to show that the linear span M of such continuous real valued functions m on G cannot be orthogonal to f on H. In fact, because of Kronecker products of representations, M is an algebra, and because the regular representation of G is faithful, M separates points of G. Thus by Stone-Weierstrass we can approximate f uniformly on H by the restrictions of functions in M. This is clearly inconsistent with f being orthogonal to the restriction of everything in M. q.e.d.

LEMMA 2. *Let G be a Lie group, H a compact subgroup, and X a Hilbert H-space. Then there exists a Hilbert G-space in which X is included as an H-kernel.*

PROOF. Let V be a Hilbert G-space in which X occurs as an H-invariant linear subspace. Let W be a Hilbert G-space in which H occurs as an isotropy group at some point w (e.g., $W = L^2(G)$ under right regular representation, f is a continuous function with compact support on G/H assuming the value one only at the coset H and $w(g) = f(gH)$). Then $S = \{(v, w) \in V \oplus W \mid v \in X\}$ is easily seen by 2.1.4 to be an H-kernel in $V \oplus W$ and $x \rightarrow (x, w)$ is an H-equivariant imbedding of X onto S. q.e.d.

LEMMA 3. *If a G-space X admits an equivariant imbedding in a Hilbert G-space then it admits one in the unit sphere of a Hilbert G-space.*

PROOF. Let $f: X \rightarrow H$ be an equivariant imbedding of X in the Hilbert G-space H and let V be a one dimensional Hilbert G-space on which G acts trivially. Then $x \rightarrow (f(x), v)/ \| (f(x), v) \|$, where v is a non-zero element of V, is an equivariant imbedding of X in the unit sphere of $H \oplus V$.

LEMMA 4. *If G is a compact Lie group then every separable, metrizable G-space X admits an equivariant imbedding in a Hilbert G-space.*

PROOF. Let $F = \{x \in X \mid G_x = G\}$. F is a closed set in X and any imbedding of F in a Hilbert space (one exists because F is separable metric) is equivariant if we simply let G act trivially. The argument of

[8, Lemma 5.2] shows that it will suffice to prove that $X-F$ admits an equivariant imbedding in a Hilbert G-space, i.e., we can assume that F is empty or, equivalently, that G_x is a proper subgroup of G for all $x \in X$. We can also assume (by induction on the dimension and number of components of G) that the theorem holds for any proper closed subgroups of G and hence for all the isotropy groups of X. Given $x_0 \in X$ let $f : S \to H$ be a G_{x_0}-equivariant imbedding of a slice S at x_0 into a Hilbert G_{x_0}-space H. By 2.1.3 and Lemmas 2 and 3 we can find a G-equivariant imbedding f^* of GS into the unit sphere of a Hilbert G-space H^*. Since X/G is completely regular we can find a continuous map $\tilde{\lambda}$ of X/G into the unit interval with $\tilde{\lambda}\,(\Pi_X(x_0)) = 1$ and $\tilde{\lambda}$ identically zero in a neighborhood of the complement of $\Pi_X(S)$. Define λ on X by $\lambda = \tilde{\lambda} \circ \Pi_X$. Then we get an equivariant map $f' : X \to H^*$ by $f'(x) = \lambda(x)f^*(x)$ if $x \in GS$ and $f'(x) = 0$ if $x \notin GS$. Moreover if $U = \lambda^{-1}((0, 1])$ then f' is clearly a homeomorphism on U. Since X is separable it is now clear that we can construct the following: a sequence $\{U_n\}$ of open invariant subspaces of X which cover X, a sequence $\{f_n : X \to H_n\}$ of equivariant maps of X into Hilbert G-spaces H_n such that $\| f_n(x) \| \leqq 1/n$ for all $x \in X$ and such that f_n is a homeomorphism on U_n, and finally a sequence $\{\lambda_n\}$ of invariant continuous real valued functions on X such that $| \lambda_n(x) | \leqq 1/n$ for all $x \in X$ and $U_n = \{x \in X \mid \lambda_n(x) \neq 0\}$. Now let H be the Hilbert-space of square summable sequences of real numbers, made into a G-space by letting G act trivially and define $\lambda : X \to H$ by $\lambda(x) = (\lambda_1(x), \lambda_2(x), \cdots, \lambda_n(x), \cdots)$. Since the λ_i are invariant, λ is equivariant. The weak continuity of λ is clear and norm continuity follows as usual from $| \lambda_n | \leqq 1/n$. For the same reason the map $f : X \to H \oplus \bigoplus_{i=1}^{\infty} H_i$ given by $x \to (\lambda(x), f_1(x), \cdots f_n(x), \cdots)$ is norm continuous and it is clearly equivariant. To complete the proof we show that if $f(x_k) \to f(x)$ then $x_k \to x$ which will show both that f is one-to-one and that f^{-1} is continuous. In fact choose a U_i containing x. Then $\lambda_i(x) \neq 0$ and since $\lambda_i(x_k) \to \lambda_i(x)$ it follows that $\lambda_i(x_k) \neq 0$ for sufficiently large k and hence that $x_k \in U_i$ for large k. But $f_i(x_k) \to f_i(x)$ and since f_i is a homeomorphism on U_i it follows that $x_k \to x$. q.e.d.

4.3.3. THEOREM. *If G is a Lie group and X is a separable, metrizable, proper G-space, then X admits an equivariant imbedding in a Hilbert G-space.*

PROOF. By Lemma 4 the theorem holds for each isotropy group of G acting on X. We can now simply repeat the proof of Lemma 4 from the point where we saw we could make the corresponding assumption in that case. q.e.d.

4.3.4. THEOREM. *Every separable, metrizable, proper G-space X*

admits an invariant metric. If ρ is an invariant metric then $\tilde{\rho}(\tilde{x}, \tilde{y}) = \operatorname{Inf}\{\rho(x, y) \mid x \in \tilde{x}, y \in \tilde{y}\}$ *is an invariant metric for* X/G *and the latter is separable. If X has dimension n at x then X/G has dimension* $n - \dim G/G_x$ *at* $\Pi_X(x)$, *hence* $\dim X/G \leqq \dim X$.

PROOF. That X has an invariant metric is a trivial consequence of 4.3.3. Since ρ is invariant

$$\begin{aligned} \rho(Gx, Gz) &= \operatorname{Inf}\{(x, gz) \mid g \in G\} \\ &\leqq \operatorname{Inf}\{\rho(x, fy) + \rho(fy, gz) \mid f, g \in G\} \\ &= \operatorname{Inf}\{\sigma(x, fy) + \rho(y, f^{-1}gz) \mid f, g \in G\} \\ &= \operatorname{Inf}\{\rho(x, fy) + \rho(y, kz) \mid f, k \in G\} \\ &= \rho(Gx, Gy) + \rho(Gy, Gz) \end{aligned}$$

Since GX is a closed subset of X, by 1.1.4, if $Gx \neq Gy$ then y is not adherent to Gx so $\rho(Gx, Gy) = \operatorname{Inf}\{\rho(gx, y) \mid g \in G\} > 0$. Thus $\tilde{\rho}$ is a metric for the set X/G. It is clear from the definition of $\tilde{\rho}$ that Π_X is distance decreasing relative to ρ and $\tilde{\rho}$ and on the other hand that Π_X maps the ε-ball about x onto the ε-ball about $\Pi_X(x)$. Thus Π_X is continuous and open relative to ρ and $\tilde{\rho}$. Since these two properties characterize the topology of X/G it follows that $\tilde{\rho}$ is a metrization of X/G. Since X is separable it is Lindelöf so its continuous image X/G is Lindelöf and hence (being metrizable) separable. Finally suppose X has dimension n at x and let S be a slice at x. Since $\Pi_X(S)$ is a neighborhood of $\tilde{x} = \Pi_X(x)$ in X/G it will suffice to prove that the dimension at $\tilde{x}$ in $\Pi_X(S)$ is $n - \dim G/G_x$. Next note that, by 2.1.2, x has a neighborhood which is homeomorphic to $U \times S$ where U is an open set in a euclidean space of dimension $= \dim G/G_x$. By a theorem of Hurewicz [3] S has dimension $n - \dim G/G_x$ at x. Now the map $G_x s \to Gs$ of S/G_x onto $\Pi_X(S)$ (which is clearly continuous and is one-to-one by 2.1.4) is a homeomorphism by the corollary of 2.1.2. Now by [10, 1.7.31] the desired dimension relation is known to hold for compact groups and in particular for G_x so $\dim \Pi_X(S)$ at $x = \dim S/G_x$ at $x = \dim S$ at $x - \dim G_x/G_x = n - \dim G/G_x - 0$. q.e.d.

4.4 Equivariant imbeddings in linear G-spaces

If G is a Lie group then by an *orbit type* of G we mean a class of conjugate closed subgroups of G. If X is a G-space and x, gx are points of the same orbit of X then $G_{gx} = gG_xg^{-1}$ so that the set of isotropy groups on a given orbit form an orbit type which we call the type of that orbit. We have the following important theorem due to Yang [11] (see also [10, 1.7.26] and [8, p. 444]).

4.4.1. THEOREM. *If H is a compact Lie group then only a finite* *number of orbit types occurs in each linear H-space.*

LEMMA. *If G is a matrix group and H and K are compact subgroups of G then the set of subgroups $\{H \cap gKg^{-1} \mid g \in G\}$ fall in a finite number of H orbit types.*

PROOF. By 3.2.1, we can find a linear G-space V in which K occurs as an isotropy group say of a point v. Now regard V as a linear H-space by restriction. Then the isotropy group at gv is clearly $H \cap G_{gv} = H \cap gKg^{-1}$. Since V as an H-space has only a finite number of orbit types altogether, by 4.4.1, the lemma follows.

4.4.2. PROPOSITION. *Let G be a matrix group, X a Cartan G-space having only finitely many orbit types and H a compact subgroup of G. Then when X is regarded as an H-space by restriction it has only finitely many orbit types.*

PROOF. By assumption we can find subgroups $K_1, \cdots, K_n$ of G which are isotropy groups at points of X such that every isotropy group of X is conjugate in G to one of the K_i. By 1.1.4, each K_i is compact. If X is now regarded as an H-space then the isotropy groups will all be of the form $H \cap G_x = H \cap gK_ig^{-1}$ and the theorem is an immediate consequence of the lemma. q.e.d.

COROLLARY 1. *Let G be a matrix group, X a G-space having only finitely many orbit types and H a compact subgroup of X. Then any H-kernel in X when regarded as an H-space has only finitely many orbit types.*

COROLLARY 2. *Let G be a matrix group, X a separable metrizable G-space of finite dimension having only finitely many orbit types and H a compact subgroup of G. Then if S is an H-kernel in X, GS admits an equivariant imbedding in a linear G-space.*

PROOF. By a theorem of Mostow [8, Theorem 6.1] (see also [10, 1.8.4]) it follows from Corollary 1 that S admits an H-equivariant imbedding in a linear H-space V. By 3.1, there is a linear G-space W which, regarded as a linear H-space by restriction, has V as an invariant linear subspace. By 3.2, there is a linear G-space U having a point u such that $G_u = H$. Then by 2.1.4, it follows easily that $\{(w, u) \in W \oplus U \mid w \in V\}$ is an H-kernel in the linear G-space $W \oplus U$ which we can identify with V. Then by 2.1.3, the H-equivariant imbedding of S in V extends uniquely to a G-equivariant imbedding of GS into $W \oplus U$. q.e.d.

LEMMA 1. *Let X be a separable metric proper G-space of dimension $n < \infty$, and H a compact subgroup of G. Let $X_{<H>} = \{x \in X \mid G_x$ is conjugate to a subgroup of $H\}$. Then $X_{<H>}$ is open in X and there exist $m = n - \dim G/H + 1$ H-slices $S_1, \cdots, S_m$ in X such that $X_{<H>} \subseteqq GS_1 \cup \cdots \cup GS_m$.*

PROOF. That $X_{<H>}$ is open in X follows immediately from Corollary 2 of 2.3.3. If $\tilde{X}_{<H>} = \Pi_X(X_{<H>})$ then it follows from 4.3.4 that $\dim \tilde{X}_{<H>} \leqq m - 1$. Moreover if $\tilde{x} \in \tilde{X}_{<H>}$ there exists $x \in \tilde{x}$ with $G_x \subseteqq H$ and by Corollary 3 of 2.3.3 there exists an H-slice S at x. Clearly then $\tilde{X}_{<H>}$ has an open covering $\{\tilde{S}_\alpha\}$ such that there exists an H-slice S_α over each $\tilde{S}_\alpha$. If $\tilde{T}_\alpha$ is an open subset of $\tilde{S}_\alpha$ then there is an H-slice T_α over $\tilde{T}_\alpha$ (for in fact the restriction of f^{S_α} to $\Pi_X^{-1}(\tilde{T}_\alpha)$ is an equivariant map into G/H with $T_\alpha = S_\alpha \cap \Pi_X^{-1}(\tilde{T}_\alpha)$ the inverse image of H). In other words if $\{\tilde{S}_\beta\}$ is any refinement of $\{S_\alpha\}$ then there is an H-slice S_β over each $\tilde{S}_\beta$. Now by [10, 1.8.2] we can find a covering $\{\tilde{S}_{i\beta}\}_{\beta \in B_i}$ $i = 1, \cdots m$ of $X_{<X>}$ refining $\{\tilde{S}_\alpha\}$ such that $\tilde{S}_{i\beta} \cap \tilde{S}_{i\beta} = \varnothing$ if $\beta \neq \beta'$. Let $S_{i\beta}$ be an H-slice over $\tilde{S}_{i\beta}$. The corresponding equivariant maps $f^{S_{i\beta}} : \Pi_X^{-1}(\tilde{S}_{i\beta}) \to G/H$ have, for fixed i, disjoint open domains, hence their union is an equivariant map $f^{S_i} : \Pi_X^{-1}(\tilde{S}_i) \to G/H$ where $\tilde{S}_i = \bigcup_{\beta \in B_i} \tilde{S}_{i\beta}$, and $S_i = \bigcup_{\beta \in B_i} S_{i\beta}$ is the inverse image of H under f^{S_i} and hence a slice over $\tilde{S}_i$. Now $GS_1 \cup \cdots \cup GS_m = \bigcup_{i=1}^m \Pi_X^{-1}(\bigcup_{\beta \in B_i} \tilde{S}_{i\beta}) = \Pi_X^{-1}(\bigcup \tilde{S}_{j\beta}) = \Pi_X^{-1}(\tilde{X}_{<H>}) = X_{<H>}$. q.e.d.

LEMMA 2. *Let X be a separable metric proper G-space and $O_1, \cdots, O_m$ invariant open sets in X each of which admits an equivariant imbedding in a linear G-space. Then $O_1 \cup \cdots \cup O_m$ admits an equivariant imbedding in a linear G-space.*

PROOF. We can suppose that the O_i cover X. Let $\tilde{O}_i = \Pi_x(O_i)$. Since X/G is metrizable (4.3.4) we can find open sets $\tilde{U}_1, \cdots, \tilde{U}_m$ covering X/G such that the closure of $\tilde{U}_i$ is included in $\tilde{O}_i$. Let $\tilde{h}_i$ be a continuous real valued function on X/G with $\tilde{h}_i \mid \tilde{U}_i \equiv 1$ and $\tilde{h}_i \mid (X/G - O_i) \equiv 0$ and define invariant real valued functions h_i on X by $h_i = \tilde{h}_i \circ \Pi_X$. If f_i is an equivariant imbedding of O_i in a linear G-space V_i then we define an equivariant map f_i^* of X into V_i which is an imbedding on $U_i = \Pi_X^{-1}(\tilde{U}_i)$ by $f_i^*(x) = h_i(x)f_i(x)$ if $x \in O_i$ and $f_1^*(x) = 0$ for $x \in O_i$. Let V_0 be R^n considered as a G-space under the trivial representation of G. Then $f_0^* : x \to (h_1(x), \cdots, h_m(x))$ is an equivariant map of X into V_0. An argument completely analogous to the final part of the proof of Lemma 4 preceding 4.3.3, shows that $f : x \to (f_0^*(x), f_1(x), \cdots, f_m(x))$ is an equivariant imbedding of X into $V_0 \oplus V_1 \oplus \cdots \oplus V_m$. q.e.d.

4.4.3. THEOREM. *Let G be a matrix group and X a separable, metrizable, proper G-space of finite dimension having only finitely many orbit types. Then X admits an equivariant imbedding in a linear G-space.*

PROOF. Let $H_1, \cdots, H_n$ be compact subgroups of G such that every isotropy group of X is conjugate to some H_i and let $X_{<H_i>} = \{x \in X \mid G_x$ is conjugate to a subgroup of $H_i\}$. Then by Lemma 1 each $X_{<H_i>}$ is open in X and clearly the $X_{<H_i>}$ cover X, so by Lemma 2 it suffices to show that each $X_{<H_i>}$ admits an equivariant imbedding in a linear G-space. But by Corollaries 1 and 2 of 4.4.2, each $X_{<H_i>}$ is a finite union of open invariant subsets which admit equivariant imbeddings in linear G-spaces, so another application of Lemma 2 completes the proof. q.e.d.

We note that the restriction that G be a matrix group is essential. For let G be any Lie group and let X be G considered as a G-space acting by left translation. Then X is clearly proper and has a single orbit type (in fact a single orbit). If $f: X \to V$ is an equivariant imbedding of X in a linear G-space then clearly the isotropy group at points of the image of X in V is the identity from which it follows that G acts effectively on V, so that by definition G is a matrix group. In another sense though, 4.4.3 is not a best possible result. Namely the requirement of finitely many orbit types is not a necessary condition for equivariant imbedding in a linear G-space. In fact in [8, p. 446] Mostow exhibits a matrix group G having a linear G-space with infinitely many orbit types.

4.5. The classification of proper G-spaces

In Chapter 2 of [10] the author has given a classification theory for G-spaces with a given orbit space, G being a compact Lie group. This classification theory is analogous to and generalizes the well-known classification of principal G-bundles over a given base space in terms of homotopy classes for mappings of the base space into a classifying space. We wish to remark here that the results proved there generalize easily to the case of non-compact Lie groups if we consider only proper G-spaces. The reason is roughly this. The basic covering homotopy theorem is proved for a given group G using only the assumption that it holds for all compact proper subgroups of G and that there exists a slice through every point of a G-space. The same proof therefore works if G is any Lie group and we consider only proper G-spaces. Secondly, the universal G-spaces constructed in [2] are easily seen to be proper G-spaces and so, by 1.3.4, the G-spaces induced by mappings into their orbit spaces (which are the classifying spaces) are all proper.

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