ON A CLASS OF TRANSFORMATION GROUPS.*

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In order to apply our rather deep understanding of the structure of Lie groups to the study of transformation groups it is natural to try to single out a class of transformation groups which are in some sense naturally Lie groups. In this paper we introduce such a class and commence their study.

In Section 1 the notion of a Lie transformation group is introduced. Roughly, these are groups H of homeomorphisms of a space X which admit a Lie group topology which is strong enough to make the evaluation mapping $(h, x) \to h(x)$ of $H \times X$ into X continuous, yet weak enough so that H gets all the one-parameter subgroups it deserves by virtue of the way it acts on X (see the definition of admissibly weak below). Such a topology is uniquely determined if it exists and our efforts are in the main concerned with the question of when it exists and how one may effectively put one's hands on it when it does. A natural candidate for this so-called Lie topology is of course the compact-open topology for H. However, if one considers the example of a dense one-parameter subgroup H of the torus X acting on X by translation, it appears that this is not the general answer. In this example if we modify the compact-open topology by adding to the open sets all their arc components (getting in this way what we call the modified compact-open topology), we get the Lie topology of H. That this is a fairly general fact is one of our main results (Theorem 5.14). The latter theorem moreover shows that the reason that the compact-open topology was not good enough in the above example is connected with the fact that H was not closed in the group of all homeomorphisms of X, relative to the compact-open topology. Theorem 5.14 also states that for a large class of interesting cases the weakness condition for a Lie topology is redundant.

The remainder of the paper is concerned with developing a certain criterion for deciding when a topological group is a Lie group and applying this criterion to derive a general necessary and sufficient condition for groups of homeomorphisms of locally compact, locally connected finite dimensional metric spaces to be Lie transformation groups. The criterion is remarkable in that local compactness is not one of the assumptions. It states in fact

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that a locally arcwise connected topological group is a Lie group provided that its compact metrizable subspaces are of bounded dimension.

1. Lie transformation groups. Let G be a topological group and X a topological space. By an *action* of G on X we mean a homomorphism $\phi\colon g \to \phi_g$ of G into the group of homeomorphisms of X such that the map $(g,x) \to \phi_g(x)$ of $G \times X$ into X is continuous. If H is a group of homeomorphisms of X, then a topology for H will be called *admissibly strong* if it renders the map $(h,x) \to h(x)$ of $H \times X \to X$ continuous. We note that we do *not* demand of an admissibly strong topology that it make H a topological group; however if H is a topological group in a given topology, then clearly that topology is admissibly strong if and only if it makes the identity map of H on itself an action of H on X. Moreover if we denote by R the additive group of real numbers then:

1.1. PROPOSITION. *Let H be a topological group whose underlying group is a group of homeomorphisms of a space X. If the topology of H is admissibly strong, then each one-parameter subgroup of H is an action of R on X.*

We shall call a topology for a group H of homeomorphisms of a space X *admissibly weak* if every action of R on X whose range is in H is a continuous map of R into H with respect to this topology. Again we note that an admissibly weak topology for H is not required to make H a topological group. However from 1.1 and the definition of admissibly weak we clearly have:

1.2. PROPOSITION. *Let H be a topological group whose underlying group is a group of homeomorphisms of a space X. If the topology for H is both admissibly weak and admissibly strong then the one-parameter subgroups of H are exactly the actions of R on X whose ranges lie in H.*

The terminology 'admissibly strong' and 'admissibly weak' is justified by the following trivial observation.

1.3. PROPOSITION. *Let H be a group of homeomorphisms of a space X. A topology for H which is stronger (weaker) than an admissibly strong (weak) topology is itself admissibly strong (weak).*

Some authors use the term admissible for topologies that we call admissibly strong. For this reason we shall not succumb to the temptation of calling admissible those topologies which are at once admissibly strong and admissibly weak.

1.4. *Definition.* Let H be a group of homeomorphisms of a space X. A *Lie topology* for H is a topology for H which is both admissiblly strong and admissibly weak and which furthermore makes H a Lie group.

The following well-known fact is an immediate consequence of the existence of canonical coordinate systems of the second kind in Lie groups.

1.5. LEMMA. *Let G and H be Lie groups and h a homomorphism of the underlying group of G into the underlying group of H. A necessary and sufficient condition for h to be continuous is that $h \circ \phi$ be a one parameter subgroup of H whenever ϕ is a one-parameter subgroup of G. In particular, if G and H have the same underlying group and the same one-parameter subgroups, they are identical.*

The following proposition follows directly from 1,2, 1.5, and the definition of a Lie topology.

1.6. PROPOSITION. *A group of homeomorphisms of a topological space admits at most one Lie topology.*

1.7. *Definition.* A group of homeomorphisms of a topological space X will be called a *Lie transformation group of X* if it admits a Lie topology.

The unique Lie topology for a Lie transformation group G will be called the Lie topology for G and properties meaningful for a Lie group when used in reference to G are to be interpreted relative to its Lie topology.

2. A theorem on arcwise connected spaces. A theorem somewhat more general than the next lemma is proved on page 115 of [6], and a still more general result is indicated in exercise 7, page 80 of [8].

2.1. LEMMA. *A partitioning of the unit interval into at most countably many disjoint closed sets is trivial, i.e., contains only one element.*

2.2. THEOREM. *A partitioning of an arcwise connected space X into at most countably many disjoint closed sets is trivial.*

Proof. Let $\{E_n\}$ be such a partitioning and let $p, q \in X$. We must show that p and q are in the same E_n. Let f be a continuous map of the unit interval into X such that $f(0) = p$ and $f(1) = q$. Applying the lemma to the partitioning $\{f^{-1}(E_n)\}$ of the unit interval we see that for some n $f^{-1}(E_n)$ is the entire unit interval. Hence $p = f(0)$ and $q = f(1)$ belong to E_n.

3. Making a topology locally arcwise connected. Most, if not all, of the results of this section are known, but they belong to the realm of folk-theorems and are apparently not easily available in the literature.

Let $(X, \mathcal{T})$ be a topological space (i.e., X a set and $\mathcal{T}$ a topology for X) and let $\mathcal{B}$ be the set of arc components of all open subspaces of $(X, \mathcal{T})$. Suppose B_1 and B_2 are elements of $\mathcal{B}$ and let B_i be an arc component of $\mathcal{O}_i \in \mathcal{T}$. Then if $p \in B_1 \cap B_2$, the arc component of p in $\mathcal{O}_1 \cap \mathcal{O}_2$, which belongs to $\mathcal{B}$, is clearly a subset of $B_1 \cap B_2$. Thus $B_1 \cap B_2$ is a union of sets from $\mathcal{B}$ and hence $\mathcal{B}$ is a base for a new topology $\mathfrak{M}(\mathcal{T})$ for X which is clearly stronger than $\mathcal{T}$.

3.1. *Definition.* We define an operation $\mathfrak{M}$ on topologies as follows: if $\mathcal{T}$ is a topology for a set X then $\mathfrak{M}(\mathcal{T})$ is the topology for X which has as a base all arc components of open subspaces of $(X, \mathcal{T})$.

The following theorem summarizes some of the most important properties of the operation $\mathfrak{M}$.

3.2. THEOREM. *Let $(X, \mathcal{T})$ be a topological space.*

(1) *If $\mathcal{T}$ satisfies the first axiom of countability, so does $\mathfrak{M}(\mathcal{T})$.*

(2) *If Z is a locally arcwise connected space and f is a function from Z into X continuous relative to the topology $\mathcal{T}$, then f is also continuous relative to $\mathfrak{M}(\mathcal{T})$. In particular $(X, \mathcal{T})$ and $(X, \mathfrak{M}(\mathcal{T}))$ have the same arcs.*

(3) *$(X, \mathfrak{M}(\mathcal{T}))$ is locally arcwise connected, and in fact $\mathfrak{M}(\mathcal{T})$ can be characterized as the weakest locally arcwise connected topology for X which is stronger than $\mathcal{T}$. Hence $\mathfrak{M}$ is idempotent.*

(4) *The components of an open subset of $(X, \mathfrak{M}(\mathcal{T}))$ are just its arc components when regarded as a subspace of $(X, \mathcal{T})$. In particular the components of $(X, \mathfrak{M}(\mathcal{T}))$ are the arc components of $(X, \mathcal{T})$.*

(5) *If X is a group and $(X, \mathcal{T})$ a topological group, then $(X, \mathfrak{M}(\mathcal{T}))$ is also a topological group and it has the same one-parameter subgroups as $(X, \mathcal{T})$.*

Proof. Given $x \in X$ and a countable base $\{\mathcal{O}_n\}$ for the $\mathcal{T}$-neighborhoods of x we get a countable base $\{\mathcal{O}_n'\}$ for the $\mathfrak{M}(\mathcal{T})$-neighborhoods of x by taking $\mathcal{O}_n'$ to be the arc component of x in $\mathcal{O}_n$ (relative to the topology $\mathcal{T}$, of course). This proves (1).

Let $\mathfrak{B}$ be the set of all arc components of open subspaces of $(X, \mathcal{T})$ so that by definition $\mathfrak{B}$ is a base for $\mathfrak{M}(\mathcal{T})$.

Suppose f is a function from the locally arcwise connected space Z into X which is continuous relative to $\mathcal{T}$. Given $B \in \mathfrak{B}$ we will show that $f^{-1}(B)$ is open in Z which will prove (2). By definition of $\mathfrak{B}$ we can choose $\mathfrak{O} \in \mathcal{T}$ such that B is an arc component of $\mathfrak{O}$. Given $p \in f^{-1}(B)$ let W be the arc component of p in $f^{-1}(\mathfrak{O})$. Then $f(W)$ is arcwise connected, included in $\mathfrak{O}$, and meets B at $f(p)$; hence $f(W) \subseteq B$. Since Z is locally arcwise connected and $f^{-1}(\mathfrak{O})$ is open, W is open. Thus a neighborhood of p is included in $f^{-1}(B)$ so $f^{-1}(B)$ is open.

Next let $B \in \mathfrak{B}$. By definition of $\mathfrak{B}$, B is arcwise connected when regarded as a subspace of $(X, \mathcal{T})$. Hence by (2) B is an arcwise connected subspace of $(X, \mathfrak{M}(\mathcal{T}))$. Thus $\mathfrak{M}(\mathcal{T})$ has a base consisting of arcwise connected sets so, by definition, $\mathfrak{M}(\mathcal{T})$ is locally arcwise connected. Since every $\mathfrak{O} \in \mathcal{T}$ is the union of its arc components and hence belongs to $\mathfrak{M}(\mathcal{T})$ it follows that $\mathfrak{M}(\mathcal{T})$ is stronger than $\mathcal{T}$. Suppose $\mathcal{T}'$ is a locally arcwise topology for X stronger than $\mathcal{T}$. Then the identity mapping f of $Z = (X, \mathcal{T}')$ into $(X, \mathcal{T})$ is continuous and hence, by (2), f is a continuous map of Z into $(X, \mathfrak{M}(\mathcal{T}))$, i.e., $\mathcal{T}'$ is stronger that $\mathfrak{M}(\mathcal{T})$. This proves (3).

If $\mathfrak{O}$ is an open subspace of $(X, \mathfrak{M}(\mathcal{T}))$ then, since $\mathfrak{M}(\mathcal{T})$ is locally arcwise connected, the components of $\mathfrak{O}$ are the same as the arc components of $\mathfrak{O}$. On the other hand, by (2), the arc components of $\mathfrak{O}$ are the same whether $\mathfrak{O}$ is regarded as a subspace of $(X, \mathcal{T})$ or $(X, \mathfrak{M}(\mathcal{T}))$. This proves (4).

Finally, suppose that X is a group and let f be the map $(x, y) \rightarrow xy^{-1}$ of $X \times X \rightarrow X$. If $(X, \mathcal{T})$ is a topological group then f is a continuous map of $(X, \mathcal{T}) \times (X, \mathcal{T}) \rightarrow (X, \mathcal{T})$ and *a fortiori* (since $\mathfrak{M}(\mathcal{T})$ is stronger than $\mathcal{T}$) f is a continuous map of $(X, \mathfrak{M}(\mathcal{T})) \times (X, \mathfrak{M}(\mathcal{T})) \rightarrow (X, \mathcal{T})$. Since by (3) $(X, \mathfrak{M}(\mathcal{T})) \times (X, \mathfrak{M}(\mathcal{T}))$ is locally arcwise connected, it follows from (2) that f is a continuous map of

$$(X, \mathfrak{M}(\mathcal{T})) \times (X, \mathfrak{M}(\mathcal{T})) \rightarrow (X, \mathfrak{M}(\mathcal{T})),$$

i.e., that $(X, \mathfrak{M}(\mathcal{T}))$ is a topological group. It also follows from (2) that $(X, \mathcal{T})$ and $(X, \mathfrak{M}(\mathcal{T}))$ have the same arcs and hence the same one-parameter subgroups. This proves (5).

3.3. *Definition.* If $\mathfrak{G} = (G, \mathcal{T})$ is a topological group, then we call $(G, \mathfrak{M}(\mathcal{T}))$ the *associated locally arcwise connected group of* $\mathfrak{G}$.

4. Weakening the topology of a Lie group.

4.1. THEOREM. *Let ϕ be a one-to-one representation of a locally compact group G satisfying the second axiom of countability onto a locally arcwise connected group H. Then ϕ^{-1} is continuous, i.e., ϕ is an isomorphism of G with H.*

Proof. Let V be a compact neighborhood of e_G, the identity of G. It will suffice to show that $\phi(V)$ is a neighborhood of e_H, the identity of H. Choose an open, symmetric neighborhood U of e_G such that $\bar{U}^2 \subseteq V$. Then $V - U$ is compact, so $\phi(V - U)$ is compact and hence that complement of $\phi(V - U)$ is a neighborhood of e_H. Let X be an arcwise connected neighborhood of e_H such that XX^{-1} does not meet $\phi(V - U)$.

Given g_1 and g_2 in $\phi^{-1}(X)$ we put $g_1 \sim g_2$ if and only in $g_1 g_2^{-1} \in \bar{U}$. Since $\bar{U}$ is a symmetric neighborhood of e_G, it follows that $\sim$ is a symmetric, reflexive relation on $\phi^{-1}(X)$. If $g_1 \sim g_2$ and $g_2 \sim g_3$ then $g_1 g_3^{-1} = (g_1 g_2^{-1})(g_2 g_3^{-1}) \in \bar{U}^2 \subseteq V$. But $\phi(g_1 g_3^{-1}) = \phi(g_1)\phi(g_3)^{-1} \in XX^{-1}$ and since XX^{-1} is disjoint from $\phi(V - U)$, it follows that $g_1 g_3^{-1} \in U \subseteq \bar{U}$ so $g_1 \sim g_3$. Hence $\sim$ is also transitive and hence is an equivalence relation on $\phi^{-1}(X)$. Let $\{g_\alpha\}$ be a complete set of representatives of $\phi^{-1}(X)$ under $\sim$, one of which we can take to be e_G. Given $g \in \phi^{-1}(X)$ we can find a g_α such that $g_\alpha \sim g$ so $g \in \bar{U} g_\alpha$. Thus $\{\bar{U} g_\alpha\}$ is a covering of $\phi^{-1}(X)$. If $g \in \bar{U} g_\alpha \cap \bar{U} g_\beta$, then $g_\alpha g^{-1} \in \bar{U}^{-1} = \bar{U}$ and $g g_\beta^{-1} \in \bar{U}$, so $g_\alpha g_\beta^{-1} \in \bar{U}^2 \subseteq V$. But $\phi(g_\alpha g_\beta^{-1}) \in XX^{-1}$ which is disjoint from $\phi(V - U)$ so $g_\alpha g_\beta^{-1} \in U \subseteq \bar{U}$ so $g_\alpha \sim g_\beta$ and $\alpha = \beta$. Hence the $\bar{U} g_\alpha$ are disjoint and therefore, since they have non-empty interiors and G satisfies the second axiom of countability, it follows that $\{\bar{U} g_\alpha\}$ is a countable set. Now since ϕ is one-to-one, $\{X \cap \phi(\bar{U} g_\alpha)\}$ is a countable disjoint covering of X. Moreover since $\bar{U}$ is closed and included in V it is compact. Hence each $\bar{U} g_\alpha$ is compact, so each $\phi(\bar{U} g_\alpha)$ is compact, so each $X \cap \phi(\bar{U} g_\alpha)$ is closed in X. Now $X \cap \phi(\bar{U} e_G)$ is not empty, and in fact contains e_H. Since X is arcwise connected it follows from (2.2) that $X \cap \phi(\bar{U}) = X$. Thus $X \subseteq \phi(\bar{U}) \subseteq \phi(\bar{U}^2) \subseteq \phi(V)$ so $\phi(V)$ is a neighborhood of e_H as was to be proved.

4.2. THEOREM. *Let $\mathfrak{G}$ be a locally arcwise connected, locally compact group satisfying the second axiom of countability. If the underlying group of $\mathfrak{G}$ is a topological group $\mathfrak{G}^*$ in a topology weaker than the topology of $\mathfrak{G}$, then $\mathfrak{G}$ is the associated locally arcwise connected group of $\mathfrak{G}^*$. In particular, $\mathfrak{G}^*$ is locally arcwise connected, then $\mathfrak{G} = \mathfrak{G}^*$. In any case the arc components of open subspaces of $\mathfrak{G}^*$ form a base for the topology of $\mathfrak{G}$ and both $\mathfrak{G}$ and $\mathfrak{G}^*$ have the same one-parameter subgroups.*

Proof. Let $\mathfrak{G}^{**}$ be the associated locally arcwise connected group of $\mathfrak{G}^{*}$. Since, by definition, the topology of $\mathfrak{G}^{**}$ is the weakest locally arcwise connected topology stronger than the topology of $\mathfrak{G}^{*}$, it follows that the topology of $\mathfrak{G}$ is stronger than the topology of $\mathfrak{G}^{**}$, i.e., the identity map ϕ is a representation of $\mathfrak{G}$ on $\mathfrak{G}^{**}$. It follows from (4.1) that ϕ is an isomorphism of $\mathfrak{G}$ with $\mathfrak{G}^{**}$, i.e. $\mathfrak{G} = \mathfrak{G}^{**}$.

4.3. COROLLARY. *If $\mathfrak{G} = (G, \mathfrak{I})$ is a Lie group satisfying the second axiom of countability, then the topology $\mathfrak{I}$ of $\mathfrak{G}$ is minimal in the set of all locally arcwise connected group topologies for G.*

5. The compact-open and modified compact-open topologies. Let X be a topological space and let $\mathfrak{H}(X)$ denote the group of all homeomorphisms of X on itself. Given subsets of X $K_1, \cdots, K_n$ and $\mathcal{O}_1, \cdots, \mathcal{O}_n$ with the K_i compact and the $\mathcal{O}_i$ open define

$$(K_1, \cdots, K_n; \mathcal{O}_1, \cdots, \mathcal{O}_n) = \{h \in \mathfrak{H}(X) : h(K_i) \subseteq \mathcal{O}_i, i = 1, \cdots, n\}.$$

The compact-open topology for $\mathfrak{H}(X)$ is by definition the topology in which sets of the above form are a basis. If H is a subgroup of $\mathfrak{H}(X)$, then the compact-open topology for H is the topology induced on H by the compact-open topology for $\mathfrak{H}(X)$; equivalently it is the topology which has as a basis sets of the form

$$(K_1, \cdots, K_n; \mathcal{O}_1, \cdots, \mathcal{O}_n)_H = \{h \in H : h(K_i) \subseteq \mathcal{O}_i, i = 1, \cdots, n\}.$$

We refer the reader to [1] for details concerning the compact-open topology (it is called the k-topology there). We will need the following facts proved in [1].

5.1. *If X is locally compact, then the compact-open topology for a group of homeomorphisms of X is admissibly strong and is weaker than any other admissibly strong topology.*

5.2. *If X is locally compact and locally connected, then every group of homeomorphisms of X is a topological group in its compact-open topology.*

Immediate from the definition of the compact-open topology is

5.3. PROPOSITION. *If H is a group of homeomorphisms of a space and G a subgroup of H, then the compact-open topology for H induces on G the compact-open topology for G.*

Another fact we will need is

5.4. PROPOSITION. *If X is a locally compact space satisfying the second axiom of countability, then the compact-open topology for any group H of homeomorphisms of X also satisfies the second axiom of countability.*

Proof. Choose a basis for the topology of X consisting of a sequence $\{\mathcal{O}_i\}$ such that each $\bar{\mathcal{O}}_i$ is compact. Then sets of the form $(\bar{\mathcal{O}}_{i_1}, \cdots, \bar{\mathcal{O}}_{i_n}; \mathcal{O}_{j_1}, \cdots, \mathcal{O}_{j_n})_H$ give a countable base for the compact-open topology for H.

If $\mathfrak{G}$ is a topological group, then the bilateral uniform structure for $\mathfrak{G}$ is that uniform structure generated by uniformities of the form $\{(g,h) \in \mathfrak{G} \times \mathfrak{G} : gh^{-1} \text{ and } g^{-1}h \in V\}$ for some neighborhood V of the indentity in $\mathfrak{G}$. Like the left and right uniform structures for $\mathfrak{G}$ the bilateral uniform structure is compatible with the topology of $\mathfrak{G}$. It has a countable base, and is hence equivalent to a metric, if and only if $\mathfrak{G}$ satisfies the first axiom of countability. Now in [1] Arens shows that if X is a locally compact, locally connected space and $\mathfrak{H}(X)$ is the group of all homeomorphisms of X made into a topological group (5.2) by giving it its compact-open topology, then $\mathfrak{H}(X)$ is complete in its bilateral uniform structure (but not generally in its left and right uniform structures). If we now assume that X satisfies the second axiom of countability and use (5.4), we get a fact mentioned in a footnote of [1].

5.5. PROPOSITION. *Let X be a locally compact, locally connected space satisfying the second axiom of countability and let H be a group of homeomorphisms of X which is closed, relative to the compact-open topology, in the group of all homeomorphism of X. Then the compact-open topology for H can be derived from a complete metric, hence H is of the second category in its compact-open topology.*

Now it is a well-known fact that a continuous one-to-one homomorphism of a locally compact topological group G satisfying the second axiom of countability onto a topological group H of the second category is necessarily bicontinuous (see, for example, Theorem XIII, page 65 of Pontrjagin's *Topological Groups*, where the proof is given under the assumption that H is locally compact, but only the consequence, that H is of the second category, is actually used). Using this result together with (5.1) and (5.5) we get:

5.6. PROPOSITION. *Let X be a locally compact, locally connected space satisfying the second axiom of countability and let H be a group of homeomorphisms of X which is closed, relative to the compact-open topology, in the group of all homeomorphisms of X. If H is a topological group in an admissibly strong, locally compact topology $\mathfrak{T}$ which satisfies the second axiom of countability, then $\mathfrak{T}$ is the compact-open topology for H.*

5.7. PROPOSITION. *Let X be a locally compact space, G a topological group, and ϕ an action of G on X whose range lies in a group H of homeomorphisms of X. Then ϕ is a continuous map of G into H when the latter is given its compact-open topology.*

Proof. Let K be the kernel of ϕ. It follows from the fact that ϕ is an action that K is closed in G. Let h be the canonical homomorphism of G on G/K and $\phi = \tilde{\phi} \circ h$ the canonical factoring of ϕ. Since h is continuous, it will suffice to show that $\tilde{\phi}$ is continuous when H is given its compact-open topology. Now it follows from the fact that h is an open mapping that $\tilde{\phi}$ is an action of G/K on X and of course $\tilde{\phi}$ is one-to-one. Thus it suffices to prove the theorem when ϕ is one-to-one (i.e., effective in the usual terminology). It is then no loss of generality to assume that G is a subgroup of H and that ϕ is the injection mapping. We can then restate the proposition as follows: if the topology of G is admissibly strong it is stronger than the topology induced on G by the compact-open topology of H. Since by (5.3) H induces the compact-open topology on G, this restatement is a consequence of (5.1).

Taking $G = R$ in (5.7) and recalling the definition of admissibly weak we have

5.8. COROLLARY. *If X is a locally compact space, then the compact-open topology for a group of homeomorphisms of X is always admissibly weak.*

5.9. *Definition.* Let G be a group of homeomorphisms of a space X. the ***modified compact-open topology*** for G is the topology resulting from applying the operation $\mathfrak{M}$ (3.1) to the compact-open topology for G. In other words it is the topology for G in which the arc components of open subspaces of G (relative to the compact-open topology) form a base.

A word of caution: since the operation $\mathfrak{M}$ does not in general commute with the operation of inducing a topology on a subspace, there is no analogue of (5.3) for the modified compact-open topology, i.e., the modified compact-open topology for a group H of homeomorphisms des not in general induce on a subgroup G of H the modified compact-open topology for G.

5.10. PROPOSITION. *If X is a locally compact space and G is a group of homeomorphisms of X, then the modified compact-open topology for G can be characterized as the weakest admissibly strong topology for G which is locally arcwise connected. Moreover, the modified compact-open topology for G is also admissibly weak.*

Proof. The first conclusion follows from (5.1) and (3.2(3)). If ϕ is an action of R on X with range in G, then by (5.7) ϕ is continuous when G is given its compact-open topology and hence by (3.2(2)) when G is given its modified compact-open topology, i.e., the modified compact-open topology for G is admissibly weak.

5.11. PROPOSITION. *Let X be a locally compact space and G a group of homeomorphisms of X. Then the modified compact-open topology is a Lie topology for G if and only if it makes G a Lie group. The same is true of the compact-open topology.*

Proof. This follows directly from the definition of a Lie topology (1.4) since we have seen (5.1, 5.8, 5.10) that both the compact-open and modified compact-open topologies are admissibly strong and admissibly weak.

5.12. PROPOSITION. *Let G be a group of homeomorphisms of a locally compact, locally connected space X. Then G is a locally arcwise connected topological group in its modified compact-open topology. Moreover, G has the same one-parameter subgroups when given either its compact-open topology or its modified compact-open topology, in fact, in each case they are exactly the actions of R on X with range in G.*

Proof. The first conclusion follows from (5.2) and (3.2(5)). Since we have seen that both the compact-open and modified compact-open topologies are admissibly weak and admissibly strong, the final conclusion follows from (1.2).

5.13. PROPOSITION. *If X is a locally compact space satisfying the second axiom of countability, then the modified compact-open topology for any group of homeomorphisms of X satisfies the first axiom of countability.*

Proof. (5.4) and (3.2(1)).

In general, however, it will not be true in the above case that the modified compact-open topology, like the compact-open topology, satisfies the second axiom of countability.

The following is one of our main results concerning Lie transformation groups.

5.14. THEOREM. *Let $\mathfrak{G} = (G, \mathfrak{T})$ be a Lie group satisfying the second axiom of countability whose underlying group G is a group of homeomorphisms of a locally compact, locally connected space X. If the topology $\mathfrak{T}$ of $\mathfrak{G}$ is admissibly strong, then it is automatically admissibly weak and hence G is a Lie transformation group of X and $\mathfrak{T}$ its Lie topology. Moreover $\mathfrak{T}$*

is the modified compact-open topology for G and if X satisfies the second axiom of countability and G is closed, relative to the compact-open topology, in the group of all homeomorphisms of X, then $\mathfrak{T}$ is the compact-open topology for G.

Proof. Since by (5.10) the modified compact-open topology for G is admissibly weak, everything will follow once we show that $\mathfrak{T}$ is the modified compact-open topology (the final conclusion is a consequence of (5.6)).

Let $\mathfrak{G}^*$ denote G taken with the modified compact-open topology. By (5.12) $\mathfrak{G}^*$ is a locally arcwise connected topological group. Since $\mathfrak{G}$ is locally arcwise connected and has an admissibly strong topology, it follows from (5.10) that the topology of $\mathfrak{G}$ is stronger than the topology of $\mathfrak{G}^*$. Then by (4.3) $\mathfrak{G} = \mathfrak{G}^*$, i.e., $\mathfrak{T}$ is the modified compact-open topology for G.

5.15. COROLLARY. *If a Lie topology for a group G of homeomorphisms of a locally compact, locally connected space satisfies the second axiom of countability, then it is the modified compact-open topology for G.*

5.16. COROLLARY. *If X is a locally compact, locally connected space, then the connected Lie transformation groups of X are precisely the groups of homeomorphisms of X which are connected Lie groups in their modified compact-open topology.*

Proof. (5.11) gives one part of the equivalence and, since a connected Lie group satisfies the second axiom of countability, (5.15) gives the other.

It is not possible in (5.14) to drop the assumption that $\mathfrak{G}$ satisfies the second axiom of countability. For example, if G is a non-trivial connected Lie transformation group of a locally compact, locally connected space X, then in the discrete topology G would satisfy the hypotheses, but not the conclusion of (5.14). This shows that Theorem 9 of [1] is false as stated. The latter states a result similar to part of (5.14) in a special case. It is probably true when the second axiom of countabiilty is added as an assumption on the group G, however, the simple (but invalid) analytical proof given seems irreparable and topological arguments of the type we have used seem necessary.

6. Some dimension theory. In what follows a space stated to have dimension, finite or infinite, is assumed to have a separable metric topology. This is so that all the theorems of [3] will be valid. If X is a compact space and C a closed subset of X, then $H^q(X, C)$ will denote the q-dimensional Cêch cohomology group of X modulo C, and $\Pi^q(X, C)$ will denote the q-dimensional cohomotopy classes of X relative to C, i.e., the homotopy classes of mappings of X into the q-sphere S^q which carry C into the north pole p_0. We wish first to show that if $\dim X = q > 0$, then these two sets are in one-to-one correspondence. Recall first that if $c \in X$ and $q > 0$, then $H^q(X, \{c\})$ is isomorphic to $H^q(X) = H^q(X, \emptyset)$ and that $\Pi^q(X, \{c\})$ is clearly in one-to-one correspondence with $\Pi^q(X) = \Pi^q(X, \emptyset)$. Now let $\tilde{X}$ be the space formed by identifying the closed subset C of the compact space X to a single point c, and let f be the natural projection of X on $\tilde{X}$. Then f is a relative homeomorphism of (X, C) on $(\tilde{X}, \{c\})$, so, by Theorem 5.4, pages 266 of [2], f^* is an isomorphism of $H^q(\tilde{X}, \{s\})$ with $H^q(X, C)$. It is also clear that $\Pi^q(X, \{C\})$ is in natural one-to-one correspondence with $\Pi^q(\tilde{X}, \{c\})$. Now suppose $\dim X = n < \infty$. Then $\dim(\tilde{X} - \{c\}) = \dim(X - C) \leqq n$ so, by Corollary 2, page 32 of [3], $\dim \tilde{X} \leqq n$. Now if $\dim \tilde{X} < n$, both $H^n(\tilde{X})$ and $\Pi^n(\tilde{X})$ contain just one point, while if $\dim \tilde{X} = n$, then $H^n(\tilde{X})$ and $\Pi^n(\tilde{X})$ are in one-to-one correspondence by Theorem VIII 2, page 149 of [3]. Suppose now that $n > 0$ and let us put all these one-to-one correspondences together:

$$H^n(X, C) \leftrightarrow H^n(\tilde{X}, \{c\}) \leftrightarrow H^n(\tilde{X}) \leftrightarrow \Pi^n(\tilde{X}) \leftrightarrow \Pi^n(\tilde{X}, \{c\}) \leftrightarrow \Pi^n(X, C).$$

6.1. Lemma. *If X is a compact space of dimension n $(0 < n < \infty)$, then there is a one-to-one correspondence between $H^n(X, C)$ and $\Pi^n(X, C)$.*

6.2. Theorem. *Let X be a compact space of finite dimension $n > 0$. Then there is a closed subset C of X for which there exists an essential map of the pair (X, C) into (S^n, p_0).*

Proof. Since the inessential maps of (X, C) into (S^n, p_o) are just those in the same homotopy class as the constant map $\tilde{p}_o$, it suffices to show that $\Pi^n(X, C)$ contains more than one element for some closed subset C of X. In view of the lemma it suffices to find a closed subset C of X for which $H^n(X, C) \neq 0$. But if on the contrary $H^n(X, C) = 0$ for all closed subsets C of X, then it would follow from Theorem VIII 4, page 152 of [3] that $\dim X \leqq n - 1$, a contradiction.

6.3. Borsuk's Theorem. *Let Y be a closed subspace of a space X and C a closed subspace of Y. Let f and g be homotopic mappings of (Y, C) into (S^n, p_o). If f has an extension F over X relative to S^n, then there is an extension G of g over X such that F and G are homotopic mappings of (X, C) into (S^n, p_o).*

Proof. This is stated and proved as Theorem VI 5, page 86 of [3] for the case $C=\emptyset$. However the proof given actually proves the more general relativized form stated above.

6.4. LEMMA. *Let Y be a closed subset of a space X and C a closed subset of Y. Let F be a map of (X,C) into (S^n,p_o) such that f, the restriction of F to Y, is an inessential map of (Y,C) into (S^n,p_o). Then there is a neighborhood V of Y such that the restriction of F to V is an inessential map of (V,C) into (S^n,p_o).*

Proof. Let g be the constant map $y\rightarrow p_o$ of (Y,C) into (S^n,p_o). By assumption f and g are homotopic mappings of (Y,C) into (S^n,p_o), hence by (6.3) there is an extension G of g over X such that F and G are in the same homotopy class as mappings of (X,C) into (S^n,p_o). *A fortiori* if V is any neighborhood of Y, then the restrictions of F and G to V are homotopic maps of (V,C) into (S^n,p_o), hence it suffices to find a neighborhood V of Y for which G restricted to V is an inessential map of (V,C) into (S^n,p_o). But clearly if U is any contractible neighborhood of p_o on S^n then $G^{-1}(U)=V$ works.

The following result, or at least closely related ones, are known. However the proof is short and we include it for completeness.

6.5. THEOREM. *Let C be a closed subspace of a compact space X and let f be an essential map of the pair (X,C) into (S^n,p_o). Then the family $\mathfrak{F}$ of closed subsets Y of X which include C and for which the restriction of f to Y is an essential map of (Y,C) into (S^n,p_o) contains a minimal element.*

Proof. By Zorn's lemma it suffices to show that the ordering of $\mathfrak{F}$ by inclusion is inductive, i.e., if Γ is a chain in $\mathfrak{F}$ and F is the intersection of the elements of Γ, then we must show that $F\in\mathfrak{F}$. Suppose on the contrary that $F\notin\mathfrak{F}$. Since clearly $C\subseteq F$ this means that f restricted to F is an inessential map of (F,C) into (S^n,p_o) and hence by (6.4) there is an open set V including F such that f restricted to V is an inessential map of (V,C) into (S^n,p_o). Now $\{Y-V\colon Y\in\Gamma\}$ is a chain of compact sets with empty intersection and hence $Y-V$ is empty for some $Y\in\Gamma$. But then $Y\subseteq V$ and hence the restriction of f to Y is an inessential map of (Y,C) into (S^n,p_o) so $Y\notin\mathfrak{F}$, contradicting $\Gamma\subseteq\mathfrak{F}$.

7. A criterion for Lie groups. This section contains the proof of a theorem reported by one of the authors in [4].

7.1. THEOREM. *Let G be a locally arcwise connected topological group. If X is a non-void compact, metrizable subspace of G of dimension $n < \infty$, then either X has an interior point or else there is an arc A in G such that AX is compact, metrizable and of dimension greater than n.*

Proof. We note first that if A is any arc in G, then AX is compact and metrizable. In fact, A is the continuous image of the unit interval I under some map σ so that AX is the continuous image of the compact, metrizable space $I \times X$ under the continuous map $(t,x) \rightarrow \sigma(t)x$. The desired result now follows from Satz IX, § 3, Chapter II of [7].

If $n = 0$ then either G is discrete, in which case every point of X is an interior point, or else G has a non-trivial arc A in which case AX has a non-trivial arc and therefore is of dimension greater than or equal to one.

Now suppose $n > 0$. By (6.2) we can find a closed subset C of X for which there exists an essential map f of the pair (X, C) into (S^n, p_0). By (6.5) there is a closed subset X' of X including C such that f restricted to X' is an essential map of the pair (X', C) into (S^n, p_0), but for any non-void open subset U of X' disjoint from C the restriction of f to $X' - U$ is an inessential map of $(X' - U, C)$ onto (S^n, p_0). Without loss of generality we can assume that $e \in X' - C$, for in any case this can be arranged by a translation.

Let V be an arcwise connected neighborhood of e such that $V^{-1}\bar{V}$ is disjoint from C. Suppose now that X has no interior points. Then certainly V^{-1} is not included in X', so we can find a continuous map σ of the unit interval into G such that $\sigma(0) = e$, $A =$ range of $\sigma \subseteq V$, and $\sigma(1)^{-1} \notin X'$. Then $e \notin \sigma(1)X'$, hence since X' is compact, we can find an open neighborhood U of e with $U \subseteq V$ and $\bar{U}$ disjoint from $\sigma(1)X'$. By the choice of X' the restriction of f to $X' - U$ is homotopic, as a map of the pair $(X' - U, C)$ into (S^n, p_0), to the constant mapping $x \rightarrow p_0$. By (6.3) it follows that there is a map g of the pair (X', C) into (S^n, p_0) such that $g(x) = p_0$ for $x \in X' - U$ which is homotopic to the restriction of f to X' and therefore is essential.

Now note that $AC \cup \sigma(1)X'$ is disjoint from $\bar{U}$. In fact, $\bar{U}$ was chosen disjoint from $\sigma(1)X'$ while, since $A \subseteq V$ and $V^{-1}\bar{V}$ is disjoint from C, it follows that AC is disjoint from $\bar{V}$ and *a fortiori* from $\bar{U}$. Then since $g(x) = p_0$ for $x \in X' - U$, it follows that the defining conditions $h(x) = p_0$ for $x \in AC \cup \sigma(1)X'$, $h(x) = g(x)$ for $x \in X'$ are non-contradictory and define a continuous mapping h of $X' \cup AC \cup \sigma(1)X'$ into S^n. It follows from the essentiality of g that h does not have a continuous extension over

AX' relative to S^n. In fact, if $\tilde{h}$ were such an extension of h, then $H: I \times X' \to S^n$, defined by $H(t,x) = \tilde{h}(\sigma(t)x)$ would be a homotopy of g (considered as a map of the pair (X', C) into (S^n, p_0)) with the constant map $x \to p_0$. It follows *a fortiori* that h does not have a continuous extension over AX relative to S^n. Since as we have already seen AX is metrizable, we can apply the theorems of [3]. In particular, by Theorem VI 4, page 83 of [3], the existence of a continuous mapping of a closed subspace of AX into S^n which does not admit a continuous extension over all of AX implies that $\dim(AX) > n$.

7.2. THEOREM. *A locally arcwise connected topological group G in which the compact metrizable subspaces are of bounded dimension is a Lie group.*

Proof. Let n be the least upper bound of the dimensions of the compact, metrizable subspaces of G and let X be a compact metrizable subspace of G of dimension n. By (7.1) X has an interior point g. Then $V = g^{-1}X$ is a compact n-dimensional neighborhood of the identity in G. Hence G is a locally connected, locally compact, n-dimensional topological group and, by the theorem on page 185 of [5], G is a Lie group.

7.3. COROLLARY. *If G is a topological group in which the compact, metrizable subspaces are of bounded dimension, then the associated locally arcwise connected group of G (Definition 3.3) is a Lie group.*

Proof. Since the topology of G^*, the associated locally arcwise connected group of G, is stronger than the topology of G, the compact metrizable subspaces of G^* are also compact metrizable subspaces of G and hence have bounded dimension.

7.4. COROLLARY. *Let G be a topological group in which some neighborhood of the identity admits a continuous one-to-one map into a finite dimensional metric space. Then the associated locally arcwise connected group of G is a Lie group and in particular if G is locally arcwise connected, then G is a Lie group.*

Proof. Let f be a continuous one-to-one map of a closed neighborhood V of the identity into a finite dimensional metric space X. Given a compact subspace K of G and $k \in K$, $kV \cap K$ is a compact neighborhood of k in K. The map $h \to f(k^{-1}h)$ maps $kV \cap K$ homeomorphically onto a compact subspace of X. It follows that $kV \cap K$ is of dimension less than or equal to n. Since each point of K has a neighborhood of dimension less than or equal to n, it follows that $\dim(K) \leqq n$ and (7.4) now follows from (7.3).

8. A criterion for Lie transformation groups.

8.1. *Definition.* A set Φ of homeomorphisms of a space X will be said to be *faithfully represented* by a subset F of X if $\phi \in \Phi$ and $\phi(x) = x$ for all $x \in F$ implies ϕ is the identity map of X.

8.2. THEOREM. *Let $\mathfrak{G} = (G, \mathfrak{T})$ be a Lie group whose underlying group G is a group of homeomorphism of a space X and whose topology $\mathfrak{T}$ is admissibly strong. Then some neighborhood of the identity in $\mathfrak{G}$ is faithfully repersented by a finite subset F of X. In fact, if* $\dim \mathfrak{G} = n$ *then F can be taken to have n or fewer points.*

Proof. Given a finite subset F of X, let

$$G_F = \{g \in G : g(x) = x \text{ for all } x \in F\}.$$

Each G_F is a closed subgroup of $\mathfrak{G}$ and hence a Lie group. It will suffice to prove that for some F containing n or fewer points, G_F is zero-dimensional and hence discrete. We prove this by showing that if $\dim G_F > 0$, then for some $x \in X$ we have $\dim G_{F \cup \{x\}} < \dim G_F$. In fact, let $\phi \colon t \to \phi_t$ be a non-trivial one-parameter subgroup of G_F and let t be a real number such that $\phi_t \neq e$. Choose $x \in X$ such that $\phi_t(x) \neq x$. Then $G_{F \cup \{x\}}$ is a subgroup of G_F and ϕ is a one-parameter subgroup of G_F but not of $G_{F \cup \{x\}}$.

There is a partial converse to (8.2), namely,

8.3. THEOREM. *Let $\mathfrak{G} = (G, \mathfrak{T})$ be a topological group whose underlying group G is a group of homeomorphisms of a finite dimensional metric space X and whose topology $\mathfrak{T}$ is admissibly strong. If some neighborhood of the identity in $\mathfrak{G}$ is faithfully represented by a finite subset of X, then the associated locally arcwise connected group of $\mathfrak{G}$ is a Lie group, and in particular if $\mathfrak{G}$ is locally arcwise connected, it is a Lie group.*

Proof. Let U be a neighborhood of the identity in $\mathfrak{G}$ faithfully represented by a finite subset $x_1, \cdots, x_n$ of X and choose a neighborhood V of the identity with $V^{-1}V \subseteq U$. Then $f \colon g \to (g(x_1), \cdots, g(x_n))$ is a continuous map of V into X^n. Moreover, if $f(g) = f(h)$ for $g, h \in V$ then $g^{-1}h(x_i) = g^{-1}g(x_i) = x_i$, $i = 1, 2, \cdots, n$; since $g^{-1}h \in U$, it follows that $g^{-1}h = e$ or $g = h$. Thus f is one-to-one and since $\dim X^n \leqq n \dim X < \infty$, the proposition follows from (7.4).

8.4. THEOREM. *Let X be a locally compact, locally connected, finite dimensional metric space. A necessary and sufficient condition for a group G of homeomorphisms of X to be a Lie transformation group of X with the* *modified compact-open topology as its Lie topology is that some modified compact-open neighborhood of the identity in G be faithfully represented by a finite subset of X.*

Proof. Necessity follows from (8.2). Since the modified compact-open topology for G is admissibly strong (5.10) and makes G a locally arcwise connected topological group (5.12), sufficiency follows from (8.3).

Since the modified compact-open neighborhoods of the identity are generally impossible to determine while the compact-open neighborhoods of the identity are explicitly given, the following corollary to (8.4) is a more useful criterion than the theorem itself.

8.5. COROLLARY. *Let G be a group of homeomorphisms of a locally compact, locally connected, finite dimensional metric space X. If some compact-open neighborhood of the identity in G is faithfully represented by a finite subset of X, then G is a Lie transformation group of X and the modified compact-open topology for G is its Lie topology. In particular, if G itself is faithfully represented by a finite subset of X, it is a Lie transformation group of X with the modified compact-open topology as its Lie topology.*

Proof. A compact-open neighborhood of the identity in G is *a fortiori* a modified compact-open neighborhood of the identity.

It is perhaps in order here to remark on the relevance (or rather irrelevance) of the various metrizability assumptions we have made in Section 6 and thereafter. In general these have been made to justify the use made of theorems proved in [3], where all spaces are taken to be separable metric. However, since all the spaces to which dimension arguments are directly applied are, in this paper, compact, it is possible as the referee suggests to use the Lebesque definition of dimension in terms of the minimum number of interesecting sets in small open coverings, and drop all references to metrizability. For the results in Section 6 and for Theorem 7.1 this would strengthen our results. However for the main theorem, Theorem 7.2, dropping the reference to metrizability would at least formally weaken the theorem. Using the referee's suggestion, Corollary 7.4 can be strengthened by replacing metric by compact in its statement. That this gives a really stronger result follows from the fact that a finite dimensional metric space can be imbedded in a finite dimensional compact space. In fact the referee notes that locally compact can replace metric in Corollary 7.4, provided the dimension of a locally compact space is defined as the supremum of the dimensions of its compact subspaces. This is immediate from the proof of this corollary. Having noted this last fact it follows that in Theorems 8.3 and 8.4 and in Corollary 8.5 we can drop the assumption that X is metric.

9. A conjecture. Let X be a connected manifold satisfying the second axiom of countability. Let G be a connected Lie transformation group of X, or what is the same (5.16), a group of homeomorphisms of X that is a connected Lie group in its modified compact-open topology. Let $\bar{G}$ be the closure of G, relative to the compact-open topology, in the group of all homeomorphisms of X. Then we conjecture that $\bar{G}$ is also a Lie transformation group of X and that the Lie topology of $\bar{G}$ is its compact-open topology.

If this is so, then the structure of the class of connected Lie transformation groups of X is very clear. On the one hand, there are those groups of homeomorphisms of X which are connected Lie groups in their compact-open topology, and all others are analytic subgroups of these. The validity of this structure theorem would have many interesting consequences.

If M is a differentiable manifold with X as its underlying topological manifold, then we define a Lie transformation group of M to be a Lie transformation group of X consisting entirely of differentiable homeomorphisms. The differentiable structure of M allows one to develop an infinitesimal characterization of Lie transformation groups of M in terms of vector fields on M. This is carried out by one of the authors in a recent Memoir of the American Mathematical Society [6].

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